Dynamics of dislocation densities in a bounded channel. Part I: smooth solutions to a singular coupled parabolic system.

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Abstract

We study a coupled system of two parabolic equations in one space dimension. This system is singular because of the presence of one term with the inverse of the gradient of the solution. Our system describes an approximate model of the dynamics of dislocation densities in a bounded channel submitted to an exterior applied stress. The system of equations is written on a bounded interval and requires a special attention to the boundary layer. The proof of existence and uniqueness is done under the use of two main tools: a certain comparison principle on the gradient of the solution, and a Kozono-Taniuchi parabolic type inequality.

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1 Introduction

1.1 Setting of the problem

In this paper, we are concerned in the study of the following singular parabolic system:

$$\begin{cases} \kappa_t = \varepsilon \kappa_{xx} + \dfrac{\rho_x \rho_{xx}}{\kappa_x} - \tau \rho_x & \text{on} \quad I \times (0, \infty) \\ \rho_t = (1+\varepsilon)\rho_{xx} - \tau \kappa_x & \text{on} \quad I \times (0, \infty), \end{cases} \tag{1.1}$$

with the initial conditions:

$$\kappa(x,0) = \kappa^0(x) \quad \text{and} \quad \rho(x,0) = \rho^0(x), \tag{1.2}$$

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and the boundary conditions:

$$\begin{cases} \kappa(0,.) = \kappa^0(0) \quad \text{and} \quad \kappa(1,.) = \kappa^0(1), \\ \rho(0,.) = \rho(1,.) = 0, \end{cases} \tag{1.3}$$

where

$$\varepsilon > 0, \quad \tau \neq 0,$$

are fixed constants, and

$$I := (0,1)$$

is the open and bounded interval of $\mathbb{R}$.

The goal is to show the long-time existence and uniqueness of a smooth solution of (1.1), (1.2) and (1.3). Our motivation comes from a problem of studying the dynamics of dislocation densities in a constrained channel submitted to an exterior applied stress. In fact, system (1.1) can be seen as an approximate model of the one described in [21], where the model presented in [21] reads:

$$\begin{cases} \theta^+_t = \left[\left(\dfrac{\theta^+_x - \theta^-_x}{\theta^+ + \theta^-} - \tau \right) \theta^+ \right]_x & \text{on} \quad I \times (0,T), \\ \theta^-_t = \left[- \left(\dfrac{\theta^+_x - \theta^-_x}{\theta^+ + \theta^-} - \tau \right) \theta^- \right]_x & \text{on} \quad I \times (0,T), \end{cases} \tag{1.4}$$

with τ representing the exterior stress field. System (1.4) can be deduced from (1.1), by letting $\varepsilon = 0$; spatially differentiating the resulting system; and by considering

$$\rho^\pm_x = \theta^\pm, \quad \rho = \rho^+ - \rho^-, \quad \kappa = \rho^+ + \rho^-. \tag{1.5}$$

Here θ^+ and θ^- represent the densities of the positive and negative dislocations respectively (see [33, 25] for a physical study of dislocations).

The next challenge (that will be the motivation of another work by the authors) is to show some kind of convergence of the solution $(\rho^\varepsilon, \kappa^\varepsilon)$ of (1.1) to the solution of (1.4) as $\varepsilon \to 0$.

1.2 Statement of the main result

The main result of this paper is:

Theorem 1.1 ***(Existence and uniqueness of a solution)***
Let $0 < \alpha < 1$. Let ρ^0, κ^0 satisfying:

$$\rho^0, \kappa^0 \in C^\infty(\bar{I}), \quad \rho^0(0) = \rho^0(1) = \kappa^0(0) = 0, \quad \kappa^0(1) = 1, \tag{1.6}$$

$$\begin{cases} (1+\varepsilon)\rho^0_{xx} = \tau \kappa^0_x & \textit{on} \quad \partial I \\ (1+\varepsilon)\kappa^0_{xx} = \tau \rho^0_x & \textit{on} \quad \partial I, \end{cases} \tag{1.7}$$

and

$$\min_{x\in I}\left(\kappa_x^0(x)-|\rho_x^0(x)|\right)>0. \tag{1.8}$$

Then there exists a unique global solution (ρ,κ) of system (1.1), (1.2) and (1.3) satisfying

$$(\rho,\kappa)\in C^{3+\alpha,\frac{3+\alpha}{2}}(\bar{I}\times[0,T]) \quad \textit{for any} \quad T>0, \tag{1.9}$$

and

$$(\rho,\kappa)\in C^{\infty}(\bar{I}\times[\zeta,\infty)), \quad \forall\zeta>0. \tag{1.10}$$

Moreover, this solution also satisfies :

$$\kappa_x>|\rho_x| \quad \textit{on} \quad \bar{I}\times[0,\infty). \tag{1.11}$$

Remark 1.2 *Conditions (1.7) are natural here. Indeed, the regularity (1.9) of the solution of (1.1) with the boundary conditions (1.2) and (1.3) imply in particular condition.*

Remark 1.3 *Remark that the choice $\kappa^0(0)=0$ and $\kappa^0(1)=1$ does not reduce the generality of the problem, because the problem is linear and equation (1.1) does not see the constants.*

1.3 Brief review of the literature

Parabolic problems involving singular terms have been widely studied in various aspects. Degenerate and singular parabolic equations have been extensively studied by DiBenedetto et al. (see for instance [12, 13, 14, 15, 10] and the references therein). The authors regard the solutions of singular or degenerate parabolic equations with measurable coefficients whose prototype is:

$$u_t-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)=0, \quad p>2 \text{ or } 1<p<2.$$

The study includes local Hölder continuity of bounded weak solutions, local and global boundedness of weak solutions and local intrinsic and global Harnack estimates. Other parabolic equations of the type

$$u_t-\Delta u^m=0, \quad 0<m<1,$$

are examined in [12, 16, 17]. These equations are singular at points where $u=0$. In [16], the authors investigate, for special range of m, the behavior of the solution near the points of singularity. In particular, they show that nonnegative solutions are analytic in the space variables and at least Lipschitz continuous in time. However, in [17], an intrinsic Harnack estimate for nonnegative weak solutions is established for some optimal range of the parameter m. Other class of singular parabolic equations are of the form:

$$u_t=u_{xx}+\frac{b}{x}u_x, \tag{1.12}$$

b is a certain constant. Such an equation is related to axially symmetric problems and also occurs in probability theory. A wide study of (1.12), including existence, uniqueness and representation theorems for the solution are proved (Dirichlet and Neumann boundary conditions are treated as well). In addition, differentiability and regularity properties are investigated (for the references, see [11, 37, 2, 9]). A more general form of (1.12), including semilinear equations, is treated in [32, 7, 8, 29].

An important type of equations that can be indirectly related to our system are semilinear parabolic equations:

$$u_t = \Delta u + |u|^{p-1}u, \quad p > 1. \tag{1.13}$$

Many authors have studied the blow-up phenomena for solutions of the above equation (see for instance [38, 31, 30, 22, 35, 36]). This study includes uniform estimates at the blow-up time, as well as the investigation of of upper bounds for the initial blow-up rate. Equation (1.13) can be somehow related to the first equation of (1.1), but with a singularity of the form $1/\kappa$. This can be formally seen if we first suppose that $u \geq 0$, and then we apply the following change of variables $u = 1/v$. In this case, equation (1.13) becomes:

$$v_t = \Delta v - \frac{2|\nabla v|^2}{v} - v^{2-p},$$

and hence if $p = 3$, we obtain:

$$v_t = \Delta v - \frac{1}{v}(1 + 2|\nabla v|^2). \tag{1.14}$$

Since the solution u of (1.13) may blow-up at a finite time $t = T$, then v may vanishes at $t = T$, and therefore equation (1.14) faces similar singularity to that of the first equation of (1.1), but in terms of the solution itself.

1.4 Strategy of the proof

The existence and uniqueness is made by using a fixed point argument after a slight artificial modification in the denominator κ_x of the first equation of (1.1) in order to avoid dividing by zero. We will first show the short time existence, proving in particular that

$$\kappa_x(x,t) \geq \sqrt{\gamma^2(t) + \rho_x^2(x,t)} > 0, \tag{1.15}$$

for initial conditions satisfying:

$$\kappa_x(x,0) \geq \sqrt{\gamma^2(0) + \rho_x^2(x,0)}$$

with some suitable $\gamma(t) > 0$. The only, but dangerous, inconvenience is that the function γ depends strongly on $\|\rho_{xxx}\|_\infty$, roughly speaking:

$$\gamma' \simeq -\|\rho_{xxx}\|_\infty \gamma, \tag{1.16}$$

where $\|\rho_{xxx}\|_\infty$ does not have, a priori, a good control independent of γ. Here where a logarithmic estimate interferes (see Section 2, Theorem 2.16) to obtain an upper bound of $\|\rho_{xxx}\|_\infty$ of the form

$$\|\rho_{xxx}\|_\infty \leq E\left(1 + \log^+ \frac{E}{\gamma^m}\right),$$

where E is an exponential function in time, and $m \in \mathbb{N}$. This allows, with (1.16), to have a good lower bound on γ independent of $\|\rho_{xxx}\|_\infty$. After that, due to some *a priori* estimates, we will move to show the global time existence. One key point here is that $\left|\frac{\rho_x}{\kappa_x}\right| \leq 1$ which somehow linearize the first equation of (1.1), and then allows the global existence.

1.5 Organization of the paper

This paper is organized as follows: In Section 2, we present the tools needed throughout this work; this includes a brief recall on the L^p, C^α and the BMO theory for parabolic equations. In Section 3, we show a comparison principle associated to (1.1) that will play a crucial rule in the long time existence of the solution as well as the positivity of κ_x. In section 4, we present a result of short time existence, uniqueness and regularity of a solution (ρ, κ) of an artificially modified system of (1.1). Section 5 is devoted to give some exponential bounds of the solution given in section 4. In section 6, we show a control of the $W_2^{2,1}$ norm of ρ_{xxx}. In a similar way, we show a control of the BMO norm of ρ_{xxx} in section 7. In section 8, we use a Kozono-Taniuchi parabolic type inequality to control the L^∞ norm of ρ_{xxx}. Thanks to this L^∞ control, we will improve the comparison principle of section 3. In Section 9, we prove our main result: Theorem 1.1. Finally, sections 10, 11 are appendices where we present the proofs of some standard results.

2 Tools: theory of parabolic equations

We start with some basic notations and terminology.

Abridged notation.

- I_T is the cylinder $I \times (0,T)$; $\bar{I}$ is the closure of I; $\overline{I_T}$ is the closure of I_T; ∂I is the boundary of I.

- $\|.\|_{L^p(X)} = \|.\|_{p,X}$, X is a Banach space, $p \geq 1$.

- S_T is the lateral boundary of I_T, or more precisely, $S_T = \partial I \times (0,T)$.

- $\partial^p I_T$ is the parabolic boundary of I_T, i.e. $\partial^p I_T = \overline{S_T} \cup (I \times \{t = 0\})$.

- $D_y^s u = \frac{\partial^s u}{\partial y^s}$, u is a function depending on the parameter y, $s \in \mathbb{N}$.

- $[l]$ is the floor part of $l \in \mathbb{R}$.

- $Q_r = Q_r(x_0, t_0)$ is the lower parabolic cylinder given by:
$$Q_r = \{(x,t);\ |x - x_0| < r,\ t_0 - r^2 < t < t_0\},\ r > 0,\ (x_0, t_0) \in I_T.$$

- $|\Omega|$ is the n-dimentional Lebesgue measure of the open set $\Omega \subset \mathbb{R}^n$.

- $m_\Omega(u) = \frac{1}{|\Omega|}\int_\Omega u$ is the average integral of the $u \in L^1(\Omega)$ over $\Omega \subset \mathbb{R}^n$.

2.1 L^p and C^α theory of parabolic equations

A major part of this work deals with the following typical problem in parabolic theory:

$$\begin{cases} u_t = \varepsilon u_{xx} + f & \text{on} \quad I_T \\ u(x,0) = \phi & \text{on} \quad I \\ u = \Phi & \text{on} \quad \partial I \times (0,T), \end{cases} \tag{2.1}$$

where $T > 0$ and $\varepsilon > 0$. A wide literature on the existence and uniqueness of solutions of (2.1) in different function spaces could be found for instance in [27], [20] and [28]. We will deal mainly with two types of spaces:

1. **The Sobolev space** $W_p^{2,1}(I_T)$, $1 < p < \infty$ which is the Banach space consisting of the elements in $L^p(I_T)$ having generalized derivatives of the form $D_t^r D_x^s u$, with r and s two non-negative integers satisfying the inequality $2r + s \leq 2$, also in $L^p(I_T)$. The norm in this space is defined by the equality

$$\|u\|_{W_p^{2,1}(I_T)} = \sum_{i=0}^{2} \sum_{2r+s=i} \|D_t^r D_x^s u\|_{p,I_T}.$$

2. **The Hölder spaces** $C^\ell(\bar{I})$ and $C^{\ell,\ell/2}(\overline{I_T})$, $\ell > 0$ a nonintegral positive number. The Hölder space $C^\ell(\bar{I})$ is the Banach space of all functions $v(x)$ that are continuous in $\bar{I}$, together with all derivatives up to order $[\ell]$, and have a finite norm

$$|v|_I^{(\ell)} = \langle v\rangle_I^{(\ell)} + \sum_{j=0}^{[\ell]} \langle v\rangle_I^{(j)}, \tag{2.2}$$

where

$$\langle v\rangle_I^{(0)} = |v|_I^{(0)} = \|v\|_{\infty,I},$$

$$\langle v\rangle_I^{(j)} = |D_x^j v|_I^{(0)}, \quad \langle v\rangle_I^{(\ell)} = \langle D_x^{[\ell]} v\rangle_I^{(\ell-[\ell])},$$

with

$$\langle v\rangle_I^{(\alpha)} = \inf\{c;\ |v(x) - v(x')| \leq c|x - x'|^\alpha,\ x, x' \in \bar{I}\}, \quad 0 < \alpha < 1. \tag{2.3}$$

The Hölder space $C^{\ell,\ell/2}(\overline{I_T})$ is the Banach space of functions $v(x,t)$ that are continuous in $\overline{I_T}$, together with all derivatives of the form $D_t^r D_x^s v$ for $2r + s < \ell$, and have a finite norm

$$|v|_{I_T}^{(\ell)} = \langle v\rangle_{I_T}^{(\ell)} + \sum_{j=0}^{[\ell]} \langle v\rangle_{I_T}^{(j)}, \tag{2.4}$$

where

$$\langle v\rangle_{I_T}^{(0)} = |v|_{I_T}^{(0)} = \|v\|_{\infty,I_T},$$

$$\langle v\rangle_{I_T}^{(j)} = \sum_{2r+s=j} |D_t^r D_x^s v|_{I_T}^{(0)},$$

$$\langle v\rangle_{I_T}^{(\ell)} = \langle v\rangle_{x,I_T}^{(\ell)} + \langle v\rangle_{t,I_T}^{(\ell/2)},$$

and

$$\langle v\rangle_{x,I_T}^{(\ell)} = \sum_{2r+s=[\ell]} \langle D_t^r D_x^s v\rangle_{x,I_T}^{(\ell-[\ell])}, \tag{2.5}$$

$$\langle v\rangle_{t,I_T}^{(\ell/2)} = \sum_{0<\ell-2r-s<2} \langle D_t^r D_x^s v\rangle_{t,I_T}^{\left(\frac{\ell-2r-s}{2}\right)} \tag{2.6}$$

with

$$\langle v\rangle_{x,I_T}^{(\alpha)} = \inf\{c;\ |v(x,t)-v(x',t)| \le c|x-x'|^\alpha,\ (x,t),(x',t) \in \overline{I_T}\}, \quad 0<\alpha<1, \tag{2.7}$$

$$\langle v\rangle_{t,I_T}^{(\alpha)} = \inf\{c;\ |v(x,t)-v(x,t')| \le c|t-t'|^\alpha,\ (x,t),(x,t') \in \overline{I_T}\}, \quad 0<\alpha<1. \tag{2.8}$$

The above definitions could be found in [27, Section 1]. Now, we write down the compatibilty conditions of order 0 and 1. These compatibility conditions concern the given data ϕ, Φ and f of problem (2.1).

Compatibility condition of order 0. Let $\phi \in C(\bar{I})$ and $\Phi \in C(\overline{S_T})$. We say that the compatibility condition of order 0 is satisfied if

$$\phi\big|_{\partial I} = \Phi\big|_{t=0}. \tag{2.9}$$

Compatibility condition of order 1. Let $\phi \in C^2(\bar{I})$, $\Phi \in C^1(\overline{S_T})$ and $f \in C(\overline{I_T})$. We say that the compatibility condition of order 1 is satisfied if (2.9) is satisfied and in addition we have:

$$(\varepsilon\phi_{xx} + f)\big|_{\partial I} = \frac{\partial \Phi}{\partial t}\Big|_{t=0}. \tag{2.10}$$

We state two results of existence and uniqueness adapted to our special problem. We begin by presenting the solvability of parabolic equations in Hölder spaces.

Theorem 2.1 ***(Solvability in Hölder spaces, [27, Theorem 5.2])***
Suppose $0<\alpha<2$, a non-integral number. Then for any $f \in C^{\alpha,\alpha/2}(\overline{I_T})$,

$$\phi \in C^{2+\alpha}(\bar{I}) \quad \textit{and} \quad \Phi \in C^{1+\alpha/2}(\overline{S_T}),$$

satisfying the compatibility condition of order 1 (see (2.9) and (2.10)), problem (2.1) has a unique solution $u \in C^{2+\alpha,1+\alpha/2}(\overline{I_T})$ satisfying the following inequality:

$$|u|_{I_T}^{(2+\alpha)} \le c^H \left(|f|_{I_T}^{(\alpha)} + |\phi|_I^{(2+\alpha)} + |\Phi|_{S_T}^{(1+\alpha/2)} \right), \tag{2.11}$$

for some $c^H = c^H(\varepsilon,\alpha,T) > 0$.

Remark 2.2 ***(Estimating*** $c^H(\varepsilon,\alpha,T)$***)***
The constant appearing in the above Hölder estimate (2.11) can be estimated as follows:

$$c^H(\varepsilon,\alpha,T) \le (T+1)^2 e^{c(T+1)}, \tag{2.12}$$

where $c = c(\varepsilon,\alpha) > 0$ is a positive constant. In order to obtain (2.12), we consider three cases for the time T.

Case 1, $T = 1$. In this case, we obtain $c^H(\varepsilon,\alpha,T) = c(\varepsilon,\alpha) > 1$.

Case 2, $T < 1$. We linearly extend the function Φ from $[0,T]$ to $[0,1]$, and we extend the function f from $\overline{I_T}$ to $\overline{I_1}$ by $f(x,t) = f(x,T)$ for $T \le t \le 1$. In this case, We have the same result of Case $T = 1$.

Case 3, $T > 1$. Take $n \in \mathbb{N}$ such that $n \le T \le n+1$. We obtain the estimate (2.12) on c^H by iteration. Let $F = |f|_{I_T}^{(\alpha)} + |\Phi|_{S_T}^{(1+\alpha/2)}$. We know that:

$$|u|_{I_T}^{(2+\alpha)} \le \sum_{k=1}^{n} |u|_{I\times(k-1,k)}^{(2+\alpha)} + |u|_{I\times(n,T)}^{(2+\alpha)}. \tag{2.13}$$

We use the fact that $|u(.,j)|_I^{(2+\alpha)} \le |u|_{I\times(j-1,j)}^{(2+\alpha)}$, $j \in \mathbb{N}$, and $1 \le j \le n$, we first compute for $c = c(\varepsilon,\alpha)$ given in Case 1:

$$\begin{aligned}
|u|_{I\times(n,T)}^{(2+\alpha)} &\le \sum_{i=1}^{n+1} c^i F + c^{n+1}|\phi|_I^{(2+\alpha)} \\
&\le (n+1)c^{n+1}\left(F + |\phi|_I^{(2+\alpha)}\right),
\end{aligned}$$

where for the last line, we have used the fact that $c > 1$. The other terms of (2.13) can be estimated in a similar way. Since $n+1 \le T+1$, the estimate (2.12) directly follows.

We now present the solvability in Sobolev spaces. Recall the norm of fractional Sobolev spaces. If $f \in W_p^s(a,b)$, $s > 0$ and $1 < p < \infty$, then

$$\|f\|_{W_p^s(a,b)} = \|f\|_{W_p^{[s]}(a,b)} + \left(\int_a^b \int_a^b \frac{|f^{([s])}(x) - f^{([s])}(y)|^p}{|x-y|^{1+(s-[s])p}}\right)^{1/p}. \tag{2.14}$$

Theorem 2.3 ***(Solvability in Sobolev spaces, [27, Theorem 9.1])***
Let $p > 1$, $\varepsilon > 0$ and $T > 0$. For any $f \in L^p(I_T)$,

$$\phi \in W_p^{2-2/p}(I) \quad \text{and} \quad \Phi \in W_p^{1-1/2p}(S_T), \tag{2.15}$$

*with $p \ne 3/2$ ($p = 3/2$ is called the **singular** index) satisfying in the case $p > 3/2$ the compatibility condition of order zero (see (2.9)), there exists a unique solution $u \in W_p^{2,1}(I_T)$ of (2.1) satisfying the following estimate:*

$$\|u\|_{W_p^{2,1}(I_T)} \le c\left(\|f\|_{p,I_T} + \|\phi\|_{W_p^{2-2/p}(I)} + \|\Phi\|_{W_p^{1-1/2p}(S_T)}\right), \tag{2.16}$$

for some $c = c(\varepsilon,p,T) > 0$.

Remark 2.4 ***(Neumann conditions)***
An analogous theorem of Theorem 2.3 is valid for problem (2.1), but with Neumann boundary conditions

$$u_x = 0 \quad on \quad S_T.$$

The singular index in this case will be $p = 3$, see [27, Chapter 4, Section 10].

Remark 2.5 *We recall that there exists a constant $c = c(p, T) > 0$ such that if $\varphi \in W_p^{2,1}(I_T)$, $\varphi_{|_{I\times\{0\}}} = \phi$ and $\varphi_{|_{S_T}} = \Phi$, then*

$$\|\phi\|_{W_p^{2-2/p}(I\times\{0\})} + \|\Phi\|_{W_p^{1-1/2p}(S_T)} \leq c\|\varphi\|_{W_p^{2,1}(I_T)}.$$

Remark 2.6 ***(The sense of the compatibility condition stated in Theorem 2.3)***
Remark that in the case $p > 3/2$, the two functions ϕ and Φ presented in (2.15) are continuous up to the boundary, i.e. $\phi \in C(\bar{I})$ and $\Phi \in C(\overline{S_T})$. This is due to the fact that we have

$$s = 1 - 1/2p > 2/3 \quad and \quad s' = 2 - 2/p > 2/3,$$

hence

$$sp > n \quad and \quad s'p > n,$$

where $n = 1$ is the space dimension. In this case the fractional Sobolev embedding [1] gives the result, and a sense of the compatibility condition stated in Theorem 2.3 is then given.

For a better understanding of the spaces stated in the above two theorems, especially fractional Sobolev spaces, we send the reader to [1] or [27]. The dependence of the constant c of Theorem 2.3 on the variable T will be of notable importance and this what is emphasized by the next lemma.

Lemma 2.7 ***(The constant c given by (2.16): case $\phi = 0$ and $\Phi = 0$)***
Under the same hypothesis of Theorem 2.3, with

$$\phi = 0 \quad and \quad \Phi = 0,$$

the estimate (2.16) can be written as:

$$\frac{\|u\|_{p,I_T}}{T} + \frac{\|u_x\|_{p,I_T}}{\sqrt{T}} + \|u_{xx}\|_{p,I_T} + \|u_t\|_{p,I_T} \leq c\|f\|_{p,I_T}, \tag{2.17}$$

where $c = c(\varepsilon, p) > 0$ is a positive constant depending only on p and ε.

The proof of this lemma will be done in Appendix A. Moreover, We will frequently make use of the following two lemmas also depicted from [27].

Lemma 2.8 ***(Sobolev embedding in Hölder spaces, [27, Lemma 3.3])***
(i) (Case $p > 3$). For any function $u \in W_p^{2,1}(I_T)$, if $\alpha = 1 - 3/p > 0$, i.e. $p > 3$, then

$$u \in C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{I_T}), \quad and \quad |u|_{I_T}^{(1+\alpha)} \leq c\|u\|_{W_p^{2,1}(I_T)}, \ c = c(p, T) > 0. \tag{2.18}$$

However, in terms of u_x, we have that $u_x \in C^{\alpha,\alpha/2}(\overline{I_T})$ satisfying the following estimates:

$$\|u_x\|_{\infty,I_T} \le c\left\{\delta^{\alpha}(\|u_t\|_{p,I_T} + \|u_{xx}\|_{p,I_T}) + \delta^{\alpha-2}\|u\|_{p,I_T}\right\}, \quad c = c(p) > 0, \tag{2.19}$$

and

$$\langle u_x\rangle_{I_T}^{(\alpha)} \le c\left\{\|u_t\|_{p,I_T} + \|u_{xx}\|_{p,I_T} + \frac{1}{\delta^2}\|u\|_{p,I_T}\right\}, \quad c = c(p) > 0. \tag{2.20}$$

(ii) (Case $p > 3/2$). If $u \in W_p^{2,1}(I_T)$ with $p > 3/2$, then $u \in C(\overline{I_T})$, and we have the following estimate:

$$\|u\|_{\infty,I_T} \le c\left\{\delta^{2-3/p}(\|u_t\|_{p,I_T} + \|u_{xx}\|_{p,I_T}) + \delta^{-3/p}\|u\|_{p,I_T}\right\}, \quad c = c(p) > 0. \tag{2.21}$$

In both cases $\delta = \min\{1/2, \sqrt{T}\}$.

Lemma 2.9 ***(Trace of functions in $W_p^{2,1}(I_T)$, [27, Lemma 3.4])***
If $u \in W_p^{2,1}(I_T)$, $p > 1$, then for $2r + s < 2 - 2/p$, we have

$$D_t^r D_x^s u\big|_{t=0} \in W_p^{2-2r-s-2/p}(I) \tag{2.22}$$

and

$$\|u\|_{W_p^{2-2r-s-2/p}(I)} \le c(T)\|u\|_{W_p^{2,1}(I_T)}. \tag{2.23}$$

In addition, for $2r + s < 2 - 1/p$, we have

$$D_t^r D_x^s u\big|_{\overline{S_T}} \in W_p^{1-r-s/2-1/2p}(\overline{S_T}) \tag{2.24}$$

and

$$\|u\|_{W_p^{1-r-s/2-1/2p}(\overline{S_T})} \le c(T)\|u\|_{W_p^{2,1}(I_T)}. \tag{2.25}$$

A useful technical lemma will now be presented. The proof of this lemma will be done in Appendix A.

Lemma 2.10 ***(L^∞ control of the spatial derivative)***
Let $p > 3$ and let $0 < T \le 1/4$ (this condition is taken for simplification). Then for every $u \in W_p^{2,1}(I_T)$ with

$$u = 0 \quad on \quad \partial^p(I_T)$$

in the trace sense (see Lemma (2.9)), there exists a constant $c(T,p) > 0$ such that

$$\|u_x\|_{\infty,I_T} \le c(T,p)\|u\|_{W_p^{2,1}(I_T)}, \tag{2.26}$$

with

$$c(T,p) = c(p)T^{\frac{p-3}{2p}} \to 0 \quad as \quad T \to 0. \tag{2.27}$$

2.2 BMO theory for parabolic equation

A very useful tool in this paper is the limit case of the L^p theory, $1 < p < \infty$, for parabolic equations, which is the BMO theory. Roughly speaking, if the function f appearing in (2.1) is in L^p for some $1 < p < \infty$, then we expect our solution u to have u_t and u_{xx} also in L^p. This is no longer valid in the limit case, i.e. when $p = \infty$. In this case, it is shown that the solution u of the parabolic equation have u_t and u_{xx} in the parabolic/anisotropic BMO space (bounded mean oscillation) that is convenient to give its definition here.

Definition 2.11 ***(Parabolic/Anisotropic*** BMO ***spaces)***
A function $u \in L^1_{loc}(I_T)$ is said to be of bounded mean oscillation, $u \in BMO(I_T)$, if the quantity

$$\sup_{Q_r \subset I_T} \left(\frac{1}{|Q_r|} \int_{Q_r} |u - m_{Q_r}(u)| \right)$$

is finite. Here the supremum is taken over all parabolic lower cylinders Q_r (see the beginning of Section 2 for the notation).

Remark 2.12 *The parabolic $BMO(I_T)$ space, which will be refereed, for simplicity, as the $BMO(I_T)$ space, and sometimes, where there is no confusion, as BMO space, is a Banach space equipped with the norm,*

$$\|u\|_{BMO(I_T)} = \sup_{Q_r \subset I_T} \left(\frac{1}{|Q_r|} \int_{Q_r} |u - m_{Q_r}(u)| \right). \tag{2.28}$$

We move now to the two main theorems of this subsection; the BMO theory for parabolic equations, and the Kozono-Taniuchi parabolic type inequality. To be more precise, we have the following:

Theorem 2.13 ***(****BMO* ***theory for parabolic equations)***
Consider the following Cauchy problem:

$$\begin{cases} u_t = \varepsilon u_{xx} + f & \text{on} \quad \mathbb{R} \times (0,T), \\ u(x,0) = 0. \end{cases} \tag{2.29}$$

If $f \in L^\infty(\mathbb{R} \times (0,T))$ and f is a $2I$-periodic function in space, i.e.

$$f(x+2,t) = f(x,t),$$

then there exists a unique solution $u \in BMO(\mathbb{R} \times (0,T))$ of (2.29) with

$$u_t, u_{xx} \in BMO(\mathbb{R} \times (0,T)).$$

Moreover, there exists $c > 0$ independent of T such that:

$$\|u_t\|_{BMO(\mathbb{R}\times(0,T))} + \|u_{xx}\|_{BMO(\mathbb{R}\times(0,T))} \le c\big[\|f\|_{BMO(\mathbb{R}\times(0,T))} + m_{2I\times(0,T)}(|f|)\big]. \tag{2.30}$$

The proof of this theorem will be presented in Appendix B. The next theorem shows an estimate concerning parabolic BMO spaces. This estimate, which will play an essential role in our later analysis, is a sort of control of the L^∞ norm of a given function by its BMO norm and the logarithm of its norm in a certain Sobolev space. It can also be considered as the parabolic version on a bounded domain I_T of the Kozono-Taniuchi inequality (see [26]) that we recall here.

Theorem 2.14 ***(The Kozono-Taniuchi inequality in the elliptic case, [26, Theorem 1])***
Let $1 < p < \infty$ and let $s > n/p$. There is a constant $C = C(n, p, s)$ such that the estimate

$$\|f\|_{\infty,\mathbb{R}^n} \leq C\left(1 + \|f\|_{BMO(\mathbb{R}^n)}\left(1 + \log^+ \|f\|_{W^s_p(\mathbb{R}^n)}\right)\right) \tag{2.31}$$

holds for all $f \in W^s_p(\mathbb{R}^n)$.

Remark 2.15 *It is worth mentioning that the BMO norm appearing in (2.31) is the elliptic BMO norm, i.e. the one where the supremum is taken over ordinary balls*

$$B_r(X_0) = \{X \in \mathbb{R}^n;\ |X - X_0| < r\}.$$

The original type of the logarithmic Sobolev inequality was found in [5, 6] (see also [18]), where the authors investigated the relation between L^∞, W^k_r and W^s_p and proved that there holds the embedding

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C\left(1 + \log^{\frac{r-1}{r}}\left(1 + \|u\|_{W^s_p(\mathbb{R}^n)}\right)\right), \quad sp > n$$

provided $\|u\|_{W^k_r} \leq 1$ for $kr = n$. This estimate was applied to prove existence of global solutions to the nonlinear Schrödinger equation (see [5, 23]). Similar embedding for vector functions u with div $u = 0$ was investigated in [3],

$$\|\nabla u\|_{L^\infty(\mathbb{R}^n)} \leq C\left(1 + \|\text{rot } u\|_{L^\infty(\mathbb{R}^n)}\left(1 + \log^+ \|u\|_{W^{s+1}_p(\mathbb{R}^n)}\right) + \|\text{rot } u\|_{L^2(\mathbb{R}^n)}\right), \tag{2.32}$$

with $sp > n$, where they made use of this estimate to give a blow-up criterion of solutions to the Euler equations. Estimate (2.31) is an improvement of (2.32) where a sharp version of (2.31) can be found in [34].

In our work, we need to have an estimate similar to (2.31), but for the parabolic BMO space and on the bounded domain I_T. This will be essential, on one hand, to show a suitable positive lower bound of κ_x (κ given by Theorem 1.1), and on the other hand, to show the long time existence of our solution. Indeed, there is a similar inequality and this is what will be illustrated by the next theorem.

Theorem 2.16 ***(A Kozono-Taniuchi parabolic type inequality)***
Let $v \in L^\infty(I_T) \cap W^{2,1}_2(I_T)$, then there exists a constant $c = c(T) > 0$ such that the estimate

$$\|v\|_{\infty,I_T} \leq c\|v\|_{\overline{BMO}(I_T)}\left(1 + \log^+ \|v\|_{W^{2,1}_2(I_T)} + \log^+ \|v\|_{\overline{BMO}(I_T)}\right), \tag{2.33}$$

holds, with

$$\|v\|_{\overline{BMO}(I_T)} = \|v\|_{BMO(I_T)} + \|v\|_{1,I_T}.$$

This inequality is first shown over $\mathbb{R}_x \times \mathbb{R}_t$, then it is deduced over I_T (for a sketch of the proof, see Appendix B).

3 A comparison principle

Proposition 3.1 ***(A comparison principle for system (1.1))***
Let

$$(\rho, \kappa) \in \left(C^{3+\alpha, \frac{3+\alpha}{2}}\left(\overline{I_T}\right)\right)^2, \quad \textit{for some } 0 < \alpha < 1,$$

be a solution of (1.1), (1.2) and (1.3) with $\kappa_x > 0$. *Suppose that*

$$|\rho_{xxx}| \le \widetilde{c} \quad \textit{on} \quad \overline{I_T}, \tag{3.1}$$

for some constant $\widetilde{c} > 0$. *Suppose furthermore that:*

$$\alpha_0 = \min_I (\kappa_x^0 - |\rho_x^0|) > 0. \tag{3.2}$$

Then there exists a continuous non-increasing function $\gamma(t) > 0$ *such that:*

$$\kappa_x(x,t) \ge \sqrt{\gamma^2(t) + \rho_x^2(x,t)} \quad \textit{over} \quad \overline{I_T}. \tag{3.3}$$

Moreover γ *satisfies* $\gamma(t) \ge \gamma(0)e^{-(\widetilde{c}+c)t}$ *for some constants (independent of* T*):* $\gamma(0) > 0$ *only depending on* α_0, *and* $c > 0$ *only depending on* ε *and* τ.

Proof. Throughout the proof, we will extensively use the following notation:

$$G_a(y) = \sqrt{a^2 + y^2} \quad a, y \in \mathbb{R}. \tag{3.4}$$

Without loss of generality (up to a change of variables in (x,t) and a re-definition of τ), assume in the proof that

$$I = (-1, 1).$$

Define the quantity M by:

$$M(x,t) = \kappa_x(x,t) - G_{\gamma(t)}(\rho_x(x,t)), \quad (x,t) \in \overline{I_T}, \tag{3.5}$$

$\gamma(t) > 0$ is a function to be determined. The proof could be divided into five steps.

Step 1. (Partial differential inequality satisfied by M)

We do the following computations in I_T:

$$M_t = \kappa_{xt} - G_\gamma'(\rho_x)\rho_{xt} - \frac{\gamma\gamma'}{\sqrt{\gamma^2 + \rho_x^2}}, \tag{3.6}$$

$$M_x = \kappa_{xx} - G_\gamma'(\rho_x)\rho_{xx}, \quad M_{xx} = \kappa_{xxx} - G_\gamma''(\rho_x)\rho_{xx}^2 - G_\gamma'(\rho_x)\rho_{xxx}, \tag{3.7}$$

and from (1.1) we deduce that

$$\begin{cases} \kappa_{xt} = \varepsilon\kappa_{xxx} + \dfrac{\rho_{xx}^2}{\kappa_x} + \dfrac{\rho_x\rho_{xxx}}{\kappa_x} - \dfrac{\rho_x\rho_{xx}\kappa_{xx}}{\kappa_x^2} - \tau\rho_{xx}, \\ \rho_{xt} = (1+\varepsilon)\rho_{xxx} - \tau\kappa_{xx}. \end{cases} \tag{3.8}$$

We set

$$\Gamma = \frac{\gamma\gamma'}{\sqrt{\gamma^2+\rho_x^2}}.$$

From (3.6), (3.7) and (3.8), we get:

$$\begin{aligned} M_t &= \varepsilon(\kappa_{xxx} - G'_\gamma(\rho_x)\rho_{xxx}) + \left(\frac{\rho_{xx}^2}{\kappa_x} - \frac{\rho_x\rho_{xx}\kappa_{xx}}{\kappa_x^2}\right) \\ &\quad + \left(\frac{\rho_x\rho_{xxx}}{\kappa_x} - G'_\gamma(\rho_x)\rho_{xxx}\right) - \tau(\rho_{xx} - G'_\gamma(\rho_x)\kappa_{xx}) - \Gamma \\ &= \varepsilon(M_{xx} + G''_\gamma(\rho_x)\rho_{xx}^2) + \left(\frac{\rho_{xx}^2}{\kappa_x} - \frac{\rho_x\rho_{xx}}{\kappa_x^2}M_x - \frac{\rho_x\rho_{xx}^2 G'_\gamma(\rho_x)}{\kappa_x^2}\right) \\ &\quad - \frac{\rho_{xxx}G'_\gamma(\rho_x)}{\kappa_x}M - \tau(\rho_{xx} - G'_\gamma(\rho_x)\kappa_{xx}) - \Gamma, \end{aligned}$$

where we have used in the last line that $G'_\gamma(y)G_\gamma(y) = y$. Define the function F_γ by:

$$F_\gamma(y) = y - \gamma\arctan(y/\gamma),$$

we note that $F'_\gamma = (G'_\gamma)^2$ and hence we have:

$$\begin{aligned} M_t &= \varepsilon M_{xx} + \varepsilon G''_\gamma(\rho_x)\rho_{xx}^2 - \frac{\rho_x\rho_{xx}}{\kappa_x^2}M_x + \frac{\rho_{xx}^2}{\kappa_x^2}[M + G_\gamma(\rho_x) - G'_\gamma(\rho_x)\rho_x] \\ &\quad - \frac{\rho_{xxx}G'_\gamma(\rho_x)}{\kappa_x}M - \tau[F'_\gamma(\rho_x)\rho_{xx} + (1-F'_\gamma(\rho_x))\rho_{xx} - G'_\gamma(\rho_x)\kappa_{xx}] - \Gamma \\ &= \varepsilon M_{xx} + \varepsilon G''_\gamma(\rho_x)\rho_{xx}^2 - \frac{\rho_x\rho_{xx}}{\kappa_x^2}M_x + \frac{\rho_{xx}^2}{\kappa_x^2}M + \frac{\rho_{xx}^2}{\kappa_x^2}(G_\gamma(\rho_x) - G'_\gamma(\rho_x)\rho_x) \\ &\quad - \frac{\rho_{xxx}G'_\gamma(\rho_x)}{\kappa_x}M + \tau G'_\gamma(\rho_x)M_x - \tau(1-F'_\gamma(\rho_x))\rho_{xx} - \Gamma, \end{aligned}$$

therefore

$$\begin{aligned} M_t = \varepsilon M_{xx} &+ \left(\tau G'_\gamma(\rho_x) - \frac{\rho_x\rho_{xx}}{\kappa_x^2}\right)M_x + \left(\frac{\rho_{xx}^2}{\kappa_x^2} - \frac{\rho_{xxx}G'_\gamma(\rho_x)}{\kappa_x}\right)M \\ &+ \varepsilon G''_\gamma(\rho_x)\rho_{xx}^2 + \frac{\rho_{xx}^2}{\kappa_x^2}[G_\gamma(\rho_x) - G'_\gamma(\rho_x)\rho_x] - \tau(1-F'_\gamma(\rho_x))\rho_{xx} - \Gamma. \end{aligned} \tag{3.9}$$

We notice that

$$G''_\gamma(y) = \frac{\gamma^2}{(\gamma^2+y^2)^{3/2}} \quad \text{and} \quad 1 - F'_\gamma(y) = \frac{\gamma^2}{\gamma^2+y^2}.$$

Using Young's inequality $2ab \le a^2 + b^2$, we have:

$$\frac{\tau\gamma^2|\rho_{xx}|}{\gamma^2+\rho_x^2} \le \frac{\varepsilon\gamma^2\rho_{xx}^2}{(\gamma^2+\rho_x^2)^{3/2}} + \frac{\gamma^2\tau^2}{4\varepsilon\sqrt{\gamma^2+\rho_x^2}}. \tag{3.10}$$

Plugging (3.10) into (3.9), we get:

$$
\begin{aligned}
M_t \geq \varepsilon M_{xx} &+ \left(\tau G_\gamma'(\rho_x) - \frac{\rho_x \rho_{xx}}{\kappa_x^2}\right) M_x \\
&+ \left(\frac{\rho_{xx}^2}{\kappa_x^2} - \frac{\rho_{xxx} G_\gamma'(\rho_x)}{\kappa_x}\right) M - \frac{\gamma^2 \tau^2}{4\varepsilon\sqrt{\gamma^2 + \rho_x^2}} - \frac{\gamma\gamma'}{\sqrt{\gamma^2 + \rho_x^2}}.
\end{aligned} \tag{3.11}
$$

Step 2. (The boundary conditions for M)

The boundary conditions (1.3), and the PDEs of system (1.1) imply the following equalities on the boundary (using the smoothness of the solution up to the boundary),

$$
\begin{cases}
\varepsilon \kappa_{xx} + \dfrac{\rho_x \rho_{xx}}{\kappa_x} - \tau \rho_x = 0 & \text{on} \quad \partial I \times [0, T] \\
(1+\varepsilon)\rho_{xx} - \tau \kappa_x = 0 & \text{on} \quad \partial I \times [0, T].
\end{cases} \tag{3.12}
$$

In particular (3.12) implies

$$
M_x = -\frac{\tau}{1+\varepsilon} G_\gamma'(\rho_x) M \quad \text{on} \quad \partial I \times [0, T]. \tag{3.13}
$$

To deal with the boundary condition (3.13), we now introduce the following change of unknown function:

$$
\overline{M}(x,t) = \cosh(\beta x) M(x,t), \quad (x,t) \in \overline{I_T}. \tag{3.14}
$$

We calculate $\overline{M}$ on the boundary of I to get:

$$
\overline{M}_x = \left(\beta \tanh(\beta x) - \frac{\tau}{1+\varepsilon} G_\gamma'(\rho_x)\right) \overline{M} \quad \text{on} \quad \partial I \times [0, T]. \tag{3.15}
$$

We claim that it is impossible for $\overline{M}$ to have a positive minimum at the boundary of I. Indeed we have

$$
\overline{M} \text{ has a positive minimum at } x = 1 \quad \Rightarrow \quad \overline{M}_x \leq 0;
$$

$$
\overline{M} \text{ has a positive minimum at } x = -1 \quad \Rightarrow \quad \overline{M}_x \geq 0.
$$

Both cases violate the equation (3.15) in the case of the choice of β satisfying:

$$
\beta \tanh \beta \geq \frac{\tau}{1+\varepsilon}, \tag{3.16}
$$

and hence the minimum of $\overline{M}$ is attained inside the interval I. We make the following calculation inside I_T.

$$
M_t = \frac{\overline{M}_t}{\cosh(\beta x)}, \quad M_x = \frac{1}{\cosh(\beta x)} \overline{M}_x - \frac{\beta \tanh(\beta x)}{\cosh(\beta x)} \overline{M},
$$

$$
M_{xx} = \frac{1}{\cosh(\beta x)} \overline{M}_{xx} - \frac{2\beta \tanh(\beta x)}{\cosh(\beta x)} \overline{M}_x + \frac{\beta^2 (2\tanh^2(\beta x) - 1)}{\cosh(\beta x)} \overline{M}.
$$

Using the previous identities into (3.11), we obtain:

$$\overline{M}_t \geq \varepsilon \overline{M}_{xx} + \left[\tau G_\gamma^{'}(\rho_x) - \frac{\rho_x\rho_{xx}}{\kappa_x^2} - 2\beta\varepsilon\tanh(\beta x)\right]\overline{M}_x - \frac{\cosh(\beta x)\gamma^2\tau^2}{4\varepsilon\sqrt{\gamma^2+\rho_x^2}} - \frac{\cosh(\beta x)\gamma\gamma^{'}}{\sqrt{\gamma^2+\rho_x^2}}$$
$$+ \left[\frac{\rho_{xx}^2}{\kappa_x^2} - \frac{\rho_{xxx}G_\gamma^{'}(\rho_x)}{\kappa_x} - \beta\tanh(\beta x)\left(\tau G_\gamma^{'}(\rho_x) - \frac{\rho_x\rho_{xx}}{\kappa_x^2}\right) + \varepsilon\beta^2(2\tanh^2(\beta x)-1)\right]\overline{M}. \tag{3.17}$$

Step 3. (The inequality satisfied by the minimum of $\overline{M}$)

Let

$$\overline{m}(t) = \min_{x\in I}\overline{M}(x,t).$$

Since the minimum is attained inside I, and since $\overline{M}$ is regular, there exists $x_0(t) \in I$ such that $\overline{m}(t) = \overline{M}(x_0(t),t)$. We remark that we have:

$$\overline{M}_x(x_0(t),t) = 0, \quad \text{and} \quad \overline{M}_{xx}(x_0(t),t) \geq 0,$$

and hence we write down the equation satisfied by $\overline{m}$, we get (indeed in the viscosity sense):

$$\overline{m}_t \geq \overbrace{\left(\frac{\rho_{xx}^2}{\kappa_x^2} - \frac{\rho_{xxx}G_\gamma^{'}(\rho_x)}{\kappa_x} - \beta\tanh(\beta x)\left(\tau G_\gamma^{'}(\rho_x) - \frac{\rho_x\rho_{xx}}{\kappa_x^2}\right) + \varepsilon\beta^2(2\tanh^2(\beta x)-1)\right)}^{\mathrm{R}}\overline{m}$$
$$- \frac{\cosh(\beta x)\gamma^2\tau^2}{4\varepsilon\sqrt{\gamma^2+\rho_x^2}} - \frac{\cosh(\beta x)\gamma\gamma^{'}}{\sqrt{\gamma^2+\rho_x^2}} \quad \text{at} \quad x = x_0(t). \tag{3.18}$$

Step 4. (Estimate of the term R)

We turn our attention now to the term R from (3.18). By Young's inequality $2ab \leq a^2+b^2$, we have:

$$\beta\tau\tanh(\beta x)G_\gamma^{'}(\rho_x) \leq 2\varepsilon\beta^2\tanh^2(\beta x) + \frac{\tau^2}{8\varepsilon}(G_\gamma^{'}(\rho_x))^2, \tag{3.19}$$

therefore the term R satisfies:

$$R \geq \frac{\rho_{xx}^2}{\kappa_x^2} + \beta\tanh(\beta x)\frac{\rho_x\rho_{xx}}{\kappa_x^2} - \frac{\rho_{xxx}G_\gamma^{'}(\rho_x)}{\kappa_x} - \frac{\tau^2}{8\varepsilon}(G_\gamma^{'}(\rho_x))^2 - \varepsilon\beta^2. \tag{3.20}$$

Moreover, using again the identity $ab \geq -\frac{a^2}{2} - \frac{b^2}{2}$, we get

$$\beta\tanh(\beta x)\frac{\rho_x\rho_{xx}}{\kappa_x^2} = \left(\frac{\sqrt{2}\rho_{xx}}{\kappa_x}\right)\left(\frac{\beta\tanh(\beta x)}{\sqrt{2}}\frac{\rho_x}{\kappa_x}\right) \geq -\frac{\rho_{xx}^2}{\kappa_x^2} - \frac{\beta^2\tanh^2(\beta x)}{4}\frac{\rho_x^2}{\kappa_x^2},$$

and hence (3.20) implies

$$R \geq -\frac{\rho_{xxx}G_\gamma^{'}(\rho_x)}{\kappa_x} - \frac{\beta^2\tanh^2(\beta x)}{4}\frac{\rho_x^2}{\kappa_x^2} - \frac{\tau^2}{8\varepsilon}(G_\gamma^{'}(\rho_x))^2 - \varepsilon\beta^2. \tag{3.21}$$

By the hypothesis (3.2), for all $\beta \in \mathbb{R}$, there exists a unique $\eta = \eta(\beta) > 0$ satisfying:

$$\eta^2 = \min_{x\in I}[\cosh(\beta x)(\kappa_x^0(x) - \sqrt{(\rho_x^0(x))^2 + \eta^2})]. \tag{3.22}$$

Define

$$\alpha_1 = \gamma(0) = \eta(\beta), \text{ where } \beta \text{ satisfies (3.16)}. \tag{3.23}$$

From (3.22), we know that

$$\overline{m}(0) = \alpha_1^2 > 0,$$

and the continuity of $\overline{m}$ preserves its positivity at least for short time. Then, as long as $\overline{m}$ is positive, we have

$$\kappa_x \geq \sqrt{\gamma^2 + \rho_x^2}. \tag{3.24}$$

By using (3.24), (3.1), and the basic identities

$$|\tanh(x)| \leq 1 \quad \text{and} \quad |G_\gamma^{'}| \leq 1,$$

inequality (3.21) implies:

$$\begin{aligned} R &\geq -\frac{|\rho_{xxx}|}{\sqrt{\gamma^2+\rho_x^2}} - \frac{\beta^2}{4} - \frac{\tau^2}{8\varepsilon} - \varepsilon\beta^2 \\ &\geq -\frac{\widetilde{c}}{\sqrt{\gamma^2+\rho_x^2}} - \frac{\beta^2}{4} - \frac{\tau^2}{8\varepsilon} - \varepsilon\beta^2 \\ &\geq -\frac{\widetilde{c}}{\sqrt{\gamma^2+\rho_x^2}} - c_1, \end{aligned} \tag{3.25}$$

where

$$c_1 = \frac{\beta^2}{4} + \frac{\tau^2}{8\varepsilon} + \varepsilon\beta^2.$$

Step 5. (The choice of γ and conclusion)

When $\gamma^{'} \leq 0$, we deduce from (3.18) and (3.25) that

$$\overline{m}_t \geq -\left(\frac{\widetilde{c}}{\sqrt{\gamma^2+\rho_x^2}} + c_1\right)\overline{m} - \left(\frac{\tau^2\cosh\beta}{4\varepsilon}\right)\frac{\gamma^2}{\sqrt{\gamma^2+\rho_x^2}} - \frac{\gamma\gamma^{'}}{\sqrt{\gamma^2+\rho_x^2}}. \tag{3.26}$$

We remind the reader that ρ_x appearing in the previous inequality have the following form:

$$\rho_x = \rho_x(x_0(t), t),$$

where

$$\overline{m}(t) = \overline{M}(x_0(t), t), \quad x_0(t) \in I. \tag{3.27}$$

Two cases can be considered:

<u>**Case A:** $\overline{m} = \gamma^2$ smooth.</u>

Assume first that γ is C^1 (which is not the case in general). Then we plug the function $\overline{m} = \gamma^2$ in (3.26) to deduce when $\gamma^{'} \leq 0$:

$$\left(2 + \frac{1}{\sqrt{\gamma^2 + \rho_x^2}}\right)\gamma\gamma^{'} \geq -\left(\frac{\widetilde{c}}{\sqrt{\gamma^2 + \rho_x^2}} + c_1\right)\gamma^2 - c_2\frac{\gamma^2}{\sqrt{\gamma^2 + \rho_x^2}} \tag{3.28}$$

with

$$c_2 = \frac{\tau^2 \cosh \beta}{4\varepsilon}.$$

Let

$$c^* = \max(c_1, c_2), \tag{3.29}$$

inequality (3.28) implies:

$$\gamma\gamma^{'} \geq -\left[\frac{\widetilde{c} + c^*(1 + \sqrt{\gamma^2 + \rho_x^2})}{1 + 2\sqrt{\gamma^2 + \rho_x^2}}\right]\gamma^2,$$

hence

$$\gamma\gamma^{'} \geq -(\widetilde{c} + c^*)\gamma^2. \tag{3.30}$$

In other terms

$$\overline{m}_t \geq -2(\widetilde{c} + c^*)\,\overline{m}.$$

This directly implies that $\overline{m}(t) \geq \overline{m}(0)e^{-2(\widetilde{c}+c^*)t}$.

Case B: the general case.

Simply choose

$$\gamma(t) = \alpha_1 e^{-(\widetilde{c}+c^*)t}, \tag{3.31}$$

where c^* is given by (3.29), and α_1 is given by (3.23). We claim that γ^2 is a sub-solution of (3.26). Indeed, the function γ given by (3.31) is constructed in such a way that γ^2 is a sub-solution of (3.26). To see this, we remark that γ solves the equality that corresponds to the inequality (3.30) and therefore it solves (3.30) with the reverse inequality. Hence, coming back from (3.30), we can see that γ^2 is a sub-solution of (3.26). Since

$$\gamma^2(0) = \alpha_1^2 = \overline{m}(0),$$

we deduce that

$$\overline{m}(t) \geq \gamma^2(t). \tag{3.32}$$

Finally, remark that

$$\alpha_1^2 \geq \min(\kappa_x^0 - \sqrt{(\rho_x^0)^2 + \alpha_1^2}) \geq \min(\kappa_x^0 - \rho_x^0 - \alpha_1) \geq \alpha_0 - \alpha_1,$$

i.e. $\alpha_1^2 + \alpha_1 \geq \alpha_0$. If $\alpha_1 \leq 1$, then $2\alpha_1^2 \geq \alpha_0$, therefore in general

$$\alpha_1 \geq \min\left(1, \sqrt{\frac{\alpha_0}{2}}\right) =: \alpha_2. \tag{3.33}$$

Inequality (3.32) implies in particular that we have

$$\kappa_x \geq \sqrt{\rho_x^2 + \gamma^2(t)}.$$

Finally, this result is still true with $\gamma(0) = \alpha_1 = \alpha_2$. □

4 Short time existence, uniqueness, and regularity

In this section, we will prove a result of short time existence, uniqueness and regularity of a solution of problem (1.1), (1.2) and (1.3). This could be done in two steps. At the first step, we show a short time existence and uniqueness result of a truncated system of equations that will be specified later. At the second step, we show an improved regularity of this solution by a bootstrap argument.

4.1 Short-time existence and uniqueness of a truncated system

Fix a time $T_0 > 0$. Consider the following system defined on $I \times (T_0, T_0 + T)$ by:

$$\begin{cases} \kappa_t = \varepsilon \kappa_{xx} + \dfrac{\rho_{xx} T_{2M_0}(\rho_x)}{(\gamma_0/2) + (\kappa_x - \gamma_0/2)^+} - \tau \rho_x & \text{in} \quad I \times (T_0, T_0 + T) \\ \rho_t = (1+\varepsilon)\rho_{xx} - \tau \kappa_x & \text{in} \quad I \times (T_0, T_0 + T), \end{cases} \tag{4.1}$$

with $M_0 > 0$ and $\gamma_0 > 0$ are two positive constants. Here, the function $T_a(x)$, $x \in \mathbb{R}$ and $a > 0$, is called a truncation function and is given by:

$$T_a(x) = \begin{cases} a & \text{if} \quad x \geq a \\ x & \text{if} \quad |x| < a \\ -a & \text{if} \quad x \leq -a. \end{cases} \tag{4.2}$$

The initial conditions are:

$$\begin{cases} \rho(x, T_0) = \rho^{T_0}(x) & \text{in} \quad I \times \{t = T_0\} \\ \kappa(x, T_0) = \kappa^{T_0}(x) & \text{in} \quad I \times \{t = T_0\}, \end{cases} \tag{4.3}$$

and the boundary conditions:

$$\begin{cases} \rho(0,t) = \rho(1,t) = 0 & \text{for} \quad T_0 \leq t \leq T_0 + T \\ \kappa(0,t) = 0 \ \text{ and } \ \kappa(1,t) = 1 & \text{for} \quad T_0 \leq t \leq T_0 + T. \end{cases} \tag{4.4}$$

Remark 4.1 ***(The terms p and α)***
In all what follows, and unless otherwise precised, the term p is a fixed positive real number such that

$$p > 3,$$

and the term $0 < \alpha < 1$ is a fixed real number that is related to p by the following relation

$$\alpha = 1 - 3/p.$$

We write down our next proposition:

Proposition 4.2 ***(Short time existence and uniqueness)***
Let $p > 3$, and $T_0 \geq 0$. Let

$$\rho^{T_0}, \kappa^{T_0} \in C^\infty(\bar{I} \times \{T_0\}), \quad \alpha = 1 - 3/p, \tag{4.5}$$

be two given functions such that:

$$\begin{cases} \rho^{T_0}(0) = \rho^{T_0}(1) = 0 \\ \kappa^{T_0}(0) = 0 \quad \textit{and} \quad \kappa^{T_0}(1) = 1, \end{cases} \tag{4.6}$$

$$\kappa_x^{T_0} \geq \gamma_0 \quad \textit{on} \quad I \times \{t = T_0\}, \tag{4.7}$$

and

$$\|(D_x^s \rho^{T_0}, D_x^s \kappa^{T_0})\|_{\infty,I} \leq M_0 \quad \textit{on} \quad I \times \{t = T_0\}, \quad s = 1, 2, \tag{4.8}$$

where $\gamma_0 > 0$ and $M_0 > 0$ are two given positive real numbers. Then there exists

$$T = T(M_0, \gamma_0, \varepsilon, \tau, p) > 0,$$

such that the system (4.1), (4.3) and (4.4) admits a unique solution

$$(\rho, \kappa) \in (W_p^{2,1}(I \times (T_0, T_0 + T)))^2.$$

Moreover, this solution satisfies

$$\kappa_x \geq \gamma_0/2 \quad \textit{on} \quad \bar{I} \times [T_0, T_0 + T], \tag{4.9}$$

and

$$|\rho_x| \leq 2M_0 \quad \textit{on} \quad \bar{I} \times [T_0, T_0 + T]. \tag{4.10}$$

Remark 4.3 *Remark that the regularity (4.5) of the initial conditions that we have considered is somehow strange and not natural for a result of existence in the Sobolev space $W_p^{2,1}$. In fact, the regularity (4.5), which is natural in connection with the main theorem of this paper (see Theorem 1.1), was just taken for the simplification of the forthcoming announcements of our results.*

Remark 4.4 *It is worth noticing that (4.6) justifies the compatibility of zero order with the boundary conditions (4.4) (see (2.9)).*

Proof of Proposition (4.2). Let

$$I_{T_0,T} = I \times (T_0, T_0 + T) \quad \text{and} \quad Y = W_p^{2,1}(I_{T_0,T}).$$

We will prove the existence and uniqueness for T small enough using a fixed point argument. Define the application Ψ by:

$$\begin{aligned} \Psi : Y^2 &\longmapsto \quad Y^2 \\ (\hat{\rho}, \hat{\kappa}) &\longmapsto \quad \Psi(\hat{\rho}, \hat{\kappa}) = (\rho, \kappa), \end{aligned} \tag{4.11}$$

where (ρ, κ) is a solution of the following system:

$$\begin{cases} \kappa_t = \varepsilon \kappa_{xx} + \dfrac{\rho_{xx} T_{2M_0}(\hat{\rho}_x)}{(\gamma_0/2) + (\hat{\kappa}_x - \gamma_0/2)^+} - \tau \hat{\rho}_x & \text{in} \quad I_{T_0,T}, \\ \rho_t = (1 + \varepsilon)\rho_{xx} - \tau \hat{\kappa}_x & \text{in} \quad I_{T_0,T}, \end{cases} \tag{4.12}$$

with the same initial and boundary conditions given by (4.3) and (4.4) respectively. Recall that ρ^{T_0} and κ^{T_0} verify (4.6). Hence we deduce from Theorem 2.3 (using on one hand, the fact that the source terms of both equations of (4.12) are in $L^p(I_{T_0,T})$; the fact that $\rho^{T_0}, \kappa^{T_0} \in W_p^{2-2/p}(I \times \{T_0\})$ "this is a direct consequence of (4.5)", and on the other hand, the compatibility of the boundary conditions (see Remark 4.4)), the existence and uniqueness of the solution $(\rho, \kappa) \in Y^2$ of (4.12), (4.3) and (4.4). We claim that Ψ is a contraction map over some suitable closed subset of Y^2 for T small enough. Let us clarify that the constant c that will frequently appear in the proof may vary from line to line but always has the form:

$$c = c(\varepsilon, p, \tau) > 0.$$

Assume we are searching for some $T > 0$ such that

$$0 < T < 1/4.$$

The proof is divided into three steps.

Step 1. (Defining the map Ψ over a suitable subset)

Let λ be any fixed constant. Define D_λ^ρ and D_λ^κ as the two closed subsets of Y given by:

$$D_\lambda^\rho = \{u \in Y;\ \|u_x\|_{p,I_{T_0,T}} \le \lambda,\ \ u = \rho^{T_0} \text{ on } \partial^p I_{T_0,T}\} \tag{4.13}$$

and

$$D_\lambda^\kappa = \{v \in Y;\ \|v_x\|_{p,I_{T_0,T}} \le \lambda,\ \ v = \kappa^{T_0} \text{ on } \partial^p I_{T_0,T}\}. \tag{4.14}$$

We will prove that Ψ is a well defined map over $D_\lambda^\rho \times D_\lambda^\kappa$ into itself, at least for sufficiently small time T. Let $(\hat\rho, \hat\kappa) \in D_\lambda^\rho \times D_\lambda^\kappa$ and let

$$\Psi(\hat\rho, \hat\kappa) = (\rho, \kappa).$$

We use system (4.12) to write down some estimates. Take

$$\bar\rho(x,t) = \rho(x,t) - \rho^{T_0}(x) \quad \text{and} \quad \bar\kappa(x,t) = \kappa(x,t) - \kappa^{T_0}(x). \tag{4.15}$$

From (4.12), the equations satisfied by $\bar\rho$ and $\bar\kappa$ are:

$$\begin{cases} \bar\rho_t = (1+\varepsilon)\bar\rho_{xx} + (1+\varepsilon)\rho_{xx}^{T_0} - \tau\hat\kappa_x & \text{on} \quad I_{T_0,T}, \\ \bar\rho = 0 \qquad \text{on} & \qquad \partial^p I_{T_0,T}, \end{cases} \tag{4.16}$$

and

$$\begin{cases} \bar\kappa_t = \varepsilon\bar\kappa_{xx} + \dfrac{(\bar\rho_{xx} + \rho_{xx}^{T_0})T_{2M_0}(\hat\rho_x)}{(\gamma_0/2) + (\hat\kappa_x - \gamma_0/2)^+} + \varepsilon\kappa_{xx}^{T_0} - \tau\hat\rho_x & \text{on} \quad I_{T_0,T}, \\ \bar\kappa = 0 \qquad \text{on} & \qquad \partial^p I_{T_0,T}, \end{cases} \tag{4.17}$$

respectively. We use equation (4.16) together with the estimate (2.17), we obtain

$$\begin{aligned} \|\bar\rho_x\|_{p,I_{T_0,T}} &\le c\sqrt{T}\left(\|\rho_{xx}^{T_0}\|_{p,I_{T_0,T}} + \|\hat\kappa_x\|_{p,I_{T_0,T}}\right) \\ &\le c\sqrt{T}\left(T^{1/p}\|\rho_{xx}^{T_0}\|_{p,I} + \lambda\right) \\ &\le cT^{1/p}\left(M_0 + \lambda\right), \end{aligned}$$

and from (4.15), we deduce that

$$\|\rho_x\|_{p,I_{T_0,T}} \leq cT^{1/p}(\lambda + M_0). \tag{4.18}$$

Therefore, choosing T satisfying:

$$T \leq \left(\frac{\lambda}{c(\lambda + M_0)}\right)^p \tag{4.19}$$

ensures that $\|\rho_x\|_{p,I_{T_0,T}} \leq \lambda$ and hence

$$\rho \in D_\lambda^\rho.$$

In the same way, we use equation (4.17) with the estimate (2.17) to obtain

$$\begin{aligned}
\|\bar{\kappa}_x\|_{p,I_{T_0,T}} &\leq c\sqrt{T}\left[\frac{4M_0}{\gamma_0}\|\rho_{xx}\|_{p,I_{T_0,T}} + T^{1/p}\|\kappa_{xx}^{T_0}\|_{p,I} + \|\hat{\rho}_x\|_{p,I_{T_0,T}}\right] \\
&\leq c\sqrt{T}\left[\frac{4M_0}{\gamma_0}\left(T^{1/p}M_0 + \lambda\right) + T^{1/p}M_0 + \lambda\right] \\
&\leq cT^{1/p}\left[\frac{4M_0}{\gamma_0}(M_0 + \lambda) + M_0 + \lambda\right] \\
&\leq cT^{1/p}(M_0 + \lambda)\left(\frac{4M_0}{\gamma_0} + 1\right),
\end{aligned} \tag{4.20}$$

where we have used again, passing from the first to the second line, the equation (4.16) together with the estimate (2.17). Precisely, we have used that:

$$\|\bar{\rho}_{xx}\|_{p,I_{T_0,T}} \leq c\left(T^{1/p}M_0 + \lambda\right).$$

From (4.20) and (4.15), we deduce that

$$\|\kappa_x\|_{p,I_{T_0,T}} \leq cT^{1/p}(M_0 + \lambda)\left(\frac{4M_0}{\gamma_0} + 1\right). \tag{4.21}$$

In this case, choosing

$$T \leq \left(\frac{\lambda}{c(M_0 + \lambda)\left(\frac{4M_0}{\gamma_0} + 1\right)}\right)^p \tag{4.22}$$

ensures that $\|\kappa_x\|_{p,I_{T_0,T}} \leq \lambda$ and hence

$$\kappa \in D_\lambda^\kappa.$$

From (4.19) and (4.22), we deduce that for sufficiently small time T, the map Ψ is a well defined map from $D_\lambda^\rho \times D_\lambda^\kappa$ into itself.

Step 2. (Ψ is a contraction map)

Let

$$\Psi(\hat\rho,\hat\kappa)=(\rho,\kappa) \quad \text{and} \quad \Psi(\hat\rho',\hat\kappa')=(\rho',\kappa').$$

The couple $(\rho-\rho',\kappa-\kappa')$ is the solution of the following system:

$$\begin{cases} (\kappa-\kappa')_t=\varepsilon(\kappa-\kappa')_{xx}+\dfrac{\rho_{xx}T_{2M_0}(\hat\rho_x)}{(\gamma_0/2)+(\hat\kappa_x-\gamma_0/2)^+} \\ \qquad -\dfrac{\rho'_{xx}T_{2M_0}(\hat\rho'_x)}{(\gamma_0/2)+(\hat\kappa'_x-\gamma_0/2)^+}-\tau(\hat\rho-\hat\rho')_x \quad \text{in } I_{T_0,T} \\ (\rho-\rho')_t=(1+\varepsilon)(\rho-\rho')_{xx}-\tau(\hat\kappa-\hat\kappa')_x \quad \text{in } I_{T_0,T}, \end{cases} \tag{4.23}$$

with

$$(\rho-\rho',\kappa-\kappa')=(0,0) \quad \text{on} \quad \partial^p I_{T_0,T}. \tag{4.24}$$

Step 2.1. From the second equation of (4.23), and (2.17), we have:

$$\|\rho-\rho'\|_Y\le c\|(\hat\kappa-\hat\kappa')_x\|_{p,I_{T_0,T}}. \tag{4.25}$$

By the boundary conditions (4.24) and the L^p parabolic estimate (2.17), we deduce that for some $c>0$, we have:

$$\|(\hat\kappa-\hat\kappa')_x\|_{p,I_{T_0,T}}\le c\sqrt{T}\|(\hat\kappa-\hat\kappa')_t-(\hat\kappa-\hat\kappa')_{xx}\|_{p,I_{T_0,T}}\le c\sqrt{T}\|\hat\kappa-\hat\kappa'\|_Y. \tag{4.26}$$

Therefore from (4.25),

$$\|\rho-\rho'\|_Y\le c\sqrt{T}\|\hat\kappa-\hat\kappa'\|_Y, \tag{4.27}$$

Step 2.2. Let F be the function given by:

$$F=\frac{\rho_{xx}T_{2M_0}(\hat\rho_x)}{(\gamma_0/2)+(\hat\kappa_x-\gamma_0/2)^+}-\frac{\rho'_{xx}T_{2M_0}(\hat\rho'_x)}{(\gamma_0/2)+(\hat\kappa'_x-\gamma_0/2)^+}-\tau(\hat\rho-\hat\rho')_x. \tag{4.28}$$

From the first equation of (4.23) and using (2.17), we get

$$\|\kappa-\kappa'\|_Y\le c\|F\|_{p,I_{T_0,T}}, \tag{4.29}$$

The function F can be rewritten as follows:

$$\begin{aligned} F+\tau(\hat\rho-\hat\rho')_x = {} & \overbrace{\frac{T_{2M_0}(\hat\rho_x)}{(\gamma_0/2)+(\hat\kappa_x-\gamma_0/2)^+}(\rho_{xx}-\rho'_{xx})}^{A_1}+\overbrace{\frac{\rho'_{xx}(T_{2M_0}(\hat\rho_x)-T_{2M_0}(\hat\rho'_x))}{(\gamma_0/2)+(\hat\kappa_x-\gamma_0/2)^+}}^{A_2} \\ & \overbrace{+\rho'_{xx}T_{2M_0}(\hat\rho'_x)\left(\frac{1}{(\gamma_0/2)+(\hat\kappa_x-\gamma_0/2)^+}-\frac{1}{(\gamma_0/2)+(\hat\kappa'_x-\gamma_0/2)^+}\right)}^{A_3}. \end{aligned} \tag{4.30}$$

We are going to use the system (4.23), (4.24) together with the inequality (2.17) in order to estimate each term of (4.30). First, from (4.27), we have:

$$\begin{aligned} \|A_1\|_{p,I_{T_0,T}} &\le 4\frac{M_0}{\gamma_0}\|(\rho-\rho')_{xx}\|_{p,I_{T_0,T}} \\ &\le c\frac{M_0}{\gamma_0}\sqrt{T}\|\hat\kappa-\hat\kappa'\|_Y. \end{aligned} \tag{4.31}$$

For the term A_2, we proceed as follows. We apply the L^∞ control of the spatial derivative (see Lemma 2.10) to the function $\hat{\rho} - \hat{\rho}'$, we get:

$$\|(\hat{\rho} - \hat{\rho}')_x\|_{\infty, I_{T_0,T}} \leq cT^{\frac{p-3}{2p}} \|\hat{\rho} - \hat{\rho}'\|_Y. \tag{4.32}$$

For the term ρ'_{xx}, we first remark that if we let $\bar{\rho}' = \rho' - \rho^{T_0}$, this function satisfies (4.16) with $\hat{\kappa}_x$ replaced by $\hat{\kappa}'_x$, and hence we deduce that

$$\|\rho'_{xx}\|_{p, I_{T_0,T}} \leq c(M_0 + \lambda). \tag{4.33}$$

From (4.32) and (4.33), we deduce that

$$\|A_2\|_{p, I_{T_0,T}} \leq c\frac{(M_0 + \lambda)}{\gamma_0} T^{\frac{p-3}{2p}} \|\hat{\rho} - \hat{\rho}'\|_Y. \tag{4.34}$$

The term A_3 could be treated in a similar way as the term A_2, and we obtain the following estimate:

$$\|A_3\|_{p, I_{T_0,T}} \leq c\frac{M_0(\lambda + M_0)}{\gamma_0^2} T^{\frac{p-3}{2p}} \|\hat{\kappa} - \hat{\kappa}'\|_Y. \tag{4.35}$$

Also we have

$$\|(\hat{\rho} - \hat{\rho}')_x\|_{p, I_{T_0,T}} \leq c\sqrt{T}\|(\hat{\rho} - \hat{\rho}')_t - (\hat{\rho} - \hat{\rho}')_{xx}\|_{p, I_{T_0,T}} \leq c\sqrt{T}\|\hat{\rho} - \hat{\rho}'\|_Y.$$

Step 2.3. From (4.27) in Step 2.1, and (4.29) in step 2.2, we finally get:

$$\|\Psi(\hat{\rho}, \hat{\kappa}) - \Psi(\hat{\rho}', \hat{\kappa}')\|_{Y^2} \quad \leq \quad cT^{\frac{p-3}{2p}} \left[1 + \frac{M_0}{\gamma_0} + \frac{M_0 + \lambda}{\gamma_0}\left(1 + \frac{M_0}{\gamma_0}\right)\right] \|(\hat{\rho}, \hat{\kappa}) - (\hat{\rho}', \hat{\kappa}')\|_{Y^2},$$

and therefore, taking T satisfying:

$$T < \left(\frac{1}{c\left(1 + \frac{M_0}{\gamma_0}\right)\left(1 + \frac{M_0+\lambda}{\gamma_0}\right)}\right)^{\frac{2p}{p-3}}, \tag{4.36}$$

(4.19) and (4.22), we deduce that Ψ is a contraction from $D^\rho_\lambda \times D^\kappa_\lambda$ into itself.

Step 3. (Conclusion)

In order to terminate the proof, it remains to show (4.9) and (4.10), again for sufficiently small time T. In fact, this will be done by controlling the modulus of continuity in time of ρ_x and κ_x uniformly with respect to T. The time T that we will use in Step 3 is that determined by (4.19), (4.22) and (4.36), ensuring existence and uniqueness. However, additional conditions will be imposed on T so that the inequalities (4.9) and (4.10) are valid on $\bar{Q}_T$.

Step 3.1. (Controlling the quantity ρ_x)

Indeed, from estimate (2.20), we deduce that

$$\begin{aligned}\langle\bar\rho_x\rangle^{(\alpha)}_{I_{T_0},T} &\le c\left(\|\bar\rho_t\|_{p,I_{T_0},T}+\|\bar\rho_{xx}\|_{p,I_{T_0},T}+\frac{1}{T}\|\bar\rho\|_{p,I_{T_0},T}\right)\\ &\le c(M_0+\lambda),\end{aligned}$$

where for the last line we have used estimate (2.17) for equation (4.16). Hence we have

$$\langle\bar\rho_x\rangle^{(\alpha/2)}_{t,I_{T_0},T}\le c(M_0+\lambda).$$

Call $m_1=c(M_0+\lambda)$, and recall that $\bar\rho=\rho-\rho^{T_0}$, we therefore obtain

$$\langle\rho_x\rangle^{(\alpha/2)}_{t,I_{T_0},T}\le m_1. \tag{4.37}$$

From (2.8), (4.8), and (4.37), we deduce that for any $(x,t)\in\bar Q_T$, we have

$$|\rho_x(x,t)|\le m_1T^{\alpha/2}+M_0,$$

and then for

$$T\le\left(\frac{M_0}{m_1}\right)^{2/\alpha}, \tag{4.38}$$

we obtain

$$|\rho_x|\le 2M_0,\quad \forall(x,t)\in\bar Q_T.$$

Step 3.2. (Controlling the quantity κ_x)

We argue in a similar manner in order to control $\langle\kappa_x\rangle^{(\alpha/2)}_{t,I_{T_0},T}$. Again, using (4.17), (2.20) and (2.17), we deduce that

$$\langle\kappa_x\rangle^{(\alpha/2)}_{t,I_{T_0},T}\le m_2, \tag{4.39}$$

with

$$m_2=c\,(M_0+\lambda)\left(\frac{4M_0}{\gamma_0}+1\right).$$

Following the same arguments as above, we obtain that for

$$T\le\left(\frac{\gamma_0}{2m_2}\right)^{2/\alpha}, \tag{4.40}$$

we have

$$\kappa_x\ge\gamma_0/2,\quad\forall(x,t)\in\bar Q_T.$$

By choosing T verifying (4.19), (4.22), (4.36), (4.38) and (4.40), we reach the end of the proof. □

4.2 Regularity of the solution

This subsection is devoted to show that the solution of (4.1), (4.3) and (4.4) enjoys more regularity than the one indicated in Proposition 4.2. This will be done using a special bootstrap argument, together with the Hölder regularity of solutions of parabolic equations.

Remark 4.5 ***(The computations of Proposition 3.1)***
The following proposition shows that the solution of (4.1), (4.3) and (4.4) has the sufficient regularity so that the calculation of the proof of the comparison principle (Proposition 3.1) can be done.

Proposition 4.6 ***(Regularity of the solution: bootstrap argument)***
Under the same hypothesis of Proposition 4.2, let ρ^{T_0} and κ^{T_0} satisfy:

$$(1+\varepsilon)\rho^{T_0}_{xx} = \tau\kappa^{T_0}_x \quad at \quad \partial I, \tag{4.41}$$

and

$$(1+\varepsilon)\kappa^{T_0}_{xx} = \tau\rho^{T_0}_x \quad at \quad \partial I. \tag{4.42}$$

Then the unique solution $(\rho,\kappa) \in Y^2$ given by Proposition 4.2, satisfying (4.9) and (4.10), is in fact more regular. To be more precise, it satisfies:

$$\rho \in C^{3+\alpha,\frac{3+\alpha}{2}}(\bar{I}\times[T_0,T_0+T]), \quad \alpha = 1-3/p, \tag{4.43}$$

and

$$\kappa \in C^{3+\alpha,\frac{3+\alpha}{2}}(\bar{I}\times[T_0,T_0+T]), \quad \alpha = 1-3/p, \tag{4.44}$$

where T is the time given by Proposition 4.2. Moreover, we have:

$$(\rho,\kappa) \in \Big(C^\infty(I\times(T_0,T_0+T))\Big)^2, \tag{4.45}$$

precisely,

$$(\rho,\kappa) \in \Big(C^\infty[\bar{I}\times[T_0+\delta,T_0+T]]\Big)^2, \quad \forall\, 0<\delta<T. \tag{4.46}$$

Proof. Let us first indicate that since, from (4.9) and (4.10), $\kappa_x \geq \gamma_0/2$ and $|\rho_x| \leq 2M_0$, then

$$T_{2M_0}(\rho_x) = \rho_x \quad \text{and} \quad (\gamma_0/2) + (\kappa_x - \gamma_0/2)^+ = \kappa_x,$$

therefore the system (4.1) can be rewritten as:

$$\begin{cases} \kappa_t = \varepsilon\kappa_{xx} + \dfrac{\rho_x\rho_{xx}}{\kappa_x} - \tau\rho_x & \text{on} \quad I\times(T_0,T_0+T) \\ \rho_t = (1+\varepsilon)\rho_{xx} - \tau\kappa_x & \text{on} \quad I\times(T_0,T_0+T). \end{cases} \tag{4.47}$$

For the seek of simplicity, let us suppose that $T_0 = 0$. We first write system (4.47) as a two "separated" equations:

$$\begin{cases} \rho_t = (1+\varepsilon)\rho_{xx} - \tau\kappa_x & \text{on} \quad I_T = I\times(0,T) \\ \rho(x,0) = \rho^0(x) & \text{on} \quad I \\ \rho(x,t) = 0 \qquad x\in\partial I,\ t\in[0,T]. \end{cases} \tag{4.48}$$

and

$$\begin{cases} \kappa_t = \varepsilon \kappa_{xx} + \dfrac{\rho_x \rho_{xx}}{\kappa_x} - \tau \rho_x & \text{on} \quad I_T \\ \kappa(x,0) = \kappa^0(x) & \text{on} \quad I \\ \kappa(0,t) = 0 \quad \text{and} \quad \kappa(1,t) = 1, & t \in [0,T], \end{cases} \tag{4.49}$$

where we set $\rho^0 = \rho^{T_0}$ and $\kappa^0 = \kappa^{T_0}$. The proof could be divided into three steps.

Step 1. (The Hölder regularity of the solution)

Since $\kappa \in W_p^{2,1}(I_T)$, we use Lemma 2.8 to deduce that $\kappa_x \in C^{\alpha,\alpha/2}(\overline{I_T})$. From the boundary conditions of system (4.48) and form (4.41) we deduce the compatibility of order 1 for the equation (4.48). Also, we have $\rho^0 \in C^{2+\alpha}(\bar{I})$. This altogether permits using the solvability of (4.48) in Hölder spaces (see Theorem 2.1) to deduce that

$$\rho \in C^{2+\alpha,1+\alpha/2}(\overline{I_T}), \quad \alpha = 1 - 3/p, \tag{4.50}$$

in particular, we have

$$\rho, \rho_t, \rho_x, \rho_{xx} \in C^{\alpha,\alpha/2}(\overline{I_T}). \tag{4.51}$$

From (4.51) and the fact that $\kappa_x \geq \gamma_0/2$, we deduce that the source term $\frac{\rho_x \rho_{xx}}{\kappa_x} - \tau \rho_x$ of system (4.49) lies in $C^{\alpha,\alpha/2}(\overline{I_T})$. We also have, from (4.9), (4.41) and (4.42), that:

$$\begin{aligned} \varepsilon \kappa^0_{xx} + \frac{\rho^0_x \rho^0_{xx}}{\kappa^0_x} - \tau \rho^0_x \big|_{\partial I} &= \varepsilon \kappa^0_{xx} + \frac{\tau \rho^0_x \kappa^0_x}{(1+\varepsilon)\kappa^0_x} - \tau \rho^0_x \big|_{\partial I} \\ &= \frac{\varepsilon}{1+\varepsilon} \left((1+\varepsilon)\kappa^0_{xx} - \tau \rho^0_x \right) \big|_{\partial I} \\ &= 0. \end{aligned}$$

This, together with the constant boundary condition of system (4.49), ensures the compatibility of order 1, and hence we reuse Theorem 2.1 to deduce that

$$\kappa \in C^{2+\alpha,1+\alpha/2}(\overline{I_T}), \quad \alpha = 1 - 3/p. \tag{4.52}$$

Step 2. (The increment of the Hölder regularity)

From (4.52), we see that the regularity of the source term of system (4.48) is increased. In fact, now

$$\kappa_x \in C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{I_T}), \quad \alpha = 1 - 3/p. \tag{4.53}$$

However, in order to use the Hölder solvability for the system (4.48), in particular Theorem 2.1, with this new obtained regularity of the source term (4.53), we just need to check that the compatibility of the boundary conditions is not altered. Indeed, this is the case since

$$0 < 1 + \alpha < 2.$$

We also remark from (4.5) that $\rho^0 \in C^{2+(1+\alpha)}(\bar{I})$, and therefore, we can use Theorem 2.1 to deduce that

$$\rho \in C^{3+\alpha,\frac{3+\alpha}{2}}(\overline{I_T}), \tag{4.54}$$

hence (4.43) is satisfied. Similarly, as in Step 1, (4.54) increases the regularity of the source term of system (4.49) hence

$$\frac{\rho_x \rho_{xx}}{\kappa_x} - \tau \rho_x \in C^{1+\alpha, \frac{1+\alpha}{2}}(\overline{I_T}).$$

Again the compatibility between the boundary conditions of system (4.49) is unchanged, and from (4.5), we know that $\kappa^0 \in C^{2+(1+\alpha)}(\bar{I})$. Therefore, upon reusing Theorem 2.1, we deduce that

$$\kappa \in C^{3+\alpha, \frac{3+\alpha}{2}}(\overline{I_T}), \tag{4.55}$$

hence (4.44) is satisfied and the proof is done.

Step 3. (The C^∞ regularity)

At this point, we will show how to obtain more regularity of the solution (ρ, κ) away from the initial data. Remark that if we want to follow similar arguments of what was done in the previous two steps, we might think of increasing the regularity of ρ by using the Hölder solvability, Theorem 2.1, and the fact that $\kappa_x \in C^{2+\alpha, \frac{2+\alpha}{2}}(\overline{I_T})$ (see (4.55) above). In fact, this requires higher order compatibilty conditions that are not satisfied having only (4.41) and (4.42). We send the reader to [27, Chapter 4, Section 5, page 319] for the details of these compatibility conditions. To overcome this difficulty, we introduce the following function. Let $0 < \delta < T$, define the test function $\overline{\varphi}_\delta \in C^\infty[0,T]$ by:

$$\overline{\varphi}_\delta(t) = \begin{cases} 0 & \text{if} \quad 0 \le t \le \delta/3 \\ \overline{\varphi}_\delta(t) \in (0,1) & \text{if} \quad \delta/3 < t < 2\delta/3 \\ 1 & \text{if} \quad 2\delta/3 \le t \le T. \end{cases} \tag{4.56}$$

We introduce the quantities

$$\overline{\rho} = \rho \overline{\varphi}_\delta \quad \text{and} \quad \overline{\kappa} = \kappa \overline{\varphi}_\delta. \tag{4.57}$$

We can easily check that these quantities satisfy two parabolic equations with the higher order compatibility of the initial data are all satisfied. By the bootstrap argument (see Steps 1, 2 above), we get:

$$(\overline{\rho}, \overline{\kappa}) \in C^\infty(\overline{I_T}).$$

From (4.56) and (4.57), we deduce that

$$(\overline{\rho}, \overline{\kappa}) = (\rho, \kappa) \quad \text{on} \quad [2\delta/3, T],$$

hence the C^∞ regularity (4.45) and (4.46) are both satisfied. $\square$.

5 Exponential bounds

In this section, we will give some exponential bounds of the solution given by Proposition (4.2) and having the regularity shown by Proposition (4.6).

It is very important, throughout all this section, to precise our notation concerning the constants that may certainly vary from line to line. Let us mention that a constant depending on time will be denoted by $c(T)$. Those who do not depend on T will be simply denoted by c. In all other cases, we will follow the changing of the constants in a precise manner.

Proposition 5.1 ***(Exponential bound in time for*** $\|(\rho_x(.,t),\kappa_x(.,t))\|_{\infty,I}$***)***
Let

$$(\rho,\kappa)\in C^{3+\alpha,\frac{3+\alpha}{2}}(\bar I\times[0,\infty))\cap C^\infty(I\times(0,\infty))\cap C^\infty(\bar I\times[\delta,\infty)),\ \forall\delta>0,$$

be a long time solution of the following system:

$$\begin{cases}\kappa_t=\varepsilon\kappa_{xx}+\dfrac{\rho_x\rho_{xx}}{\kappa_x}-\tau\rho_x & on\quad I\times(0,\infty)\\ \rho_t=(1+\varepsilon)\rho_{xx}-\tau\kappa_x & on\quad I\times(0,\infty),\end{cases}\tag{5.1}$$

with $\rho(x,0)=\rho^0(x)$*,* $\kappa(x,0)=\kappa^0(x)$*, and the boundary conditions*

$$\rho(0,.)=\rho(1,.)=0\quad on\quad \partial I\times[0,\infty),\tag{5.2}$$

$$\kappa(0,.)=0,\quad \kappa(1,.)=1\quad on\quad \partial I\times[0,\infty).\tag{5.3}$$

Suppose furthermore that

$$B=\frac{\rho_x}{\kappa_x}\quad satisfies\quad \|B\|_\infty<1.$$

Then we have

$$\|(\rho_x(.,t),\kappa_x(.,t))\|_{\infty,I}\le ce^{ct},\tag{5.4}$$

where

$$c=c\left(\|\rho^0\|_{W_p^{2-2/p}(I)},\|\kappa^0\|_{W_p^{2-2/p}(I)}\right)\ge 1,\quad p>3.$$

Remark 5.2 ***(Improved exponential bound)***
Concerning the exponential bound (5.4), we can even get

$$\|(\rho_x(.,t),\kappa_x(.,t))\|_{\infty,I}\le cae^{ct},$$

where $a=\left(1+\|\rho^0\|_{W_p^{2-2/p}(I)}+\|\kappa^0\|_{W_p^{2-2/p}(I)}\right)$*, and* $c>0$ *is a fixed constant independent of* $\|\rho^0\|_{W_p^{2-2/p}(I)}$ *and* $\|\kappa^0\|_{W_p^{2-2/p}(I)}$ *(see the final step of the following proof). However, this result will not be used in that refined form.*

Proof of Proposition 5.1. We use the special coupling of the system (5.1) to find our *a priori* estimate. Roughly speaking, the fact that κ_x appears as a source term in the second equation of system (5.1) permits, by the L^p theory for parabolic equations, to have L^p bounds, in terms of $\|\kappa_x\|_{p,I_T}$, on ρ_x and ρ_{xx} which in their turn appear in the source terms of the first equation of (5.1) satisfied by κ. All this permits to deduce our estimates. To be more precise, let $T>0$ an arbitrarily fixed time.

Step 1. (estimating κ_x in the L^p norm)

Let κ' be the solution of the following equation:

$$\begin{cases} \kappa'_t = \kappa'_{xx} & \text{on} \quad I_T \\ \kappa' = \kappa & \text{on} \quad \partial^p I_T. \end{cases} \tag{5.5}$$

As a solution of a parabolic equation, we use the L^p parabolic estimate (2.16) to the function κ' to deduce that:

$$\|\kappa'\|_{W_p^{2,1}(I_T)} \leq c(T) \left(\|\kappa^0\|_{W_p^{2-2/p}(I)} + 1 \right), \tag{5.6}$$

where the term 1 comes from the value of $\kappa' = \kappa$ on S_T. Take

$$\bar{\kappa} = \kappa - \kappa', \tag{5.7}$$

then the system satisfied by $\bar{\kappa}$ reads:

$$\begin{cases} \bar{\kappa}_t = \bar{\kappa}_{xx} - (\kappa'_t - \varepsilon \kappa'_{xx}) + \dfrac{\rho_x \rho_{xx}}{\kappa_x} - \tau \rho_x \quad \text{on} \quad I_T \\ \bar{\kappa} = 0 \quad \text{on} \quad \partial^p I_T. \end{cases} \tag{5.8}$$

Using the special version (2.17) of the parabolic L^p estimate to the function $\bar{\kappa}$, we obtain:

$$\|\bar{\kappa}_x\|_{p,I_T} \leq c\sqrt{T} \left(\|\kappa'_t\|_{p,I_T} + \|\kappa'_{xx}\|_{p,I_T} + \|\rho_{xx}\|_{p,I_T} + \|\rho_x\|_{p,I_T} \right), \tag{5.9}$$

where we have plugged into the constant c the terms ε, τ, p and $\|B\|_\infty$. Combining (5.6), (5.7) and (5.9), we get:

$$\|\kappa_x\|_{p,I_T} \leq c(T) \left(\|\kappa^0\|_{W_p^{2-2/p}(I)} + 1 \right) + c\sqrt{T} \|\rho\|_{W_p^{2,1}(I_T)}. \tag{5.10}$$

The term $\|\rho\|_{W_p^{2,1}(I_T)}$ appearing in the previous inequality is going to be estimated in the next step.

Step 2. (estimating ρ in the $W_p^{2,1}$ norm)

As in Step 1, let ρ', $\bar{\rho}$ be the two function defined similarly as κ', $\bar{\kappa}$ respectively (see (5.5) and (5.7)). ρ' satisfies an inequality similar to (5.6) that reads:

$$\|\rho'\|_{W_p^{2,1}(I_T)} \leq c(T) \|\rho^0\|_{W_p^{2-2/p}(I)}. \tag{5.11}$$

The term 1 disappered here because $\rho' = \rho = 0$ on $\overline{S_T}$. We write the system satisfied by $\bar{\rho}$, we obtain:

$$\begin{cases} \bar{\rho}_t = (1+\varepsilon)\bar{\rho}_{xx} + ((1+\varepsilon)\rho'_{xx} - \rho'_t) - \tau \kappa_x \quad \text{on} \quad I_T \\ \bar{\rho}(x,0) = 0 \quad \text{on} \quad \partial^p I_T, \end{cases} \tag{5.12}$$

hence the following estimate on $\bar{\rho}$, due to the special L^p interior estimate (2.17), holds:

$$\|\bar{\rho}\|_{W_p^{2,1}(I_T)} \leq c\left(\|\rho_t'\|_{p,I_T} + \|\rho_{xx}'\|_{p,I_T} + \|\kappa_x\|_{p,I_T}\right). \tag{5.13}$$

Again, we have plugged ε, τ and p into the constant c, and we have assumed that $T \leq 1$. Combining (5.11) and (5.13), we get in terms of ρ:

$$\|\rho\|_{W_p^{2,1}(I_T)} \leq c(T)\|\rho^0\|_{W_p^{2-2/p}(I)} + c\|\kappa_x\|_{p,I_T}. \tag{5.14}$$

We will use this estimate in order to have a control on $\|\kappa_x\|_{p,I_T}$ for sufficently small time.

Step 3. (Estimate on a small time interval)

From (5.10) and (5.14), we deduce that:

$$\|\kappa_x\|_{p,I_T} \leq c(T)\left(\|\kappa^0\|_{W_p^{2-2/p}(I)} + \|\rho^0\|_{W_p^{2-2/p}(I)} + 1\right) + c\sqrt{T}\|\kappa_x\|_{p,I_T}. \tag{5.15}$$

Let us remind the reader that all constants c and $c(T)$ have been changing from line to line. In fact, the important thing is whether they depend on T or not. Let

$$T^* = \frac{1}{2c^2}, \quad c \text{ is the constant appearing in (5.15),}$$

we deduce, from (5.15), that

$$\|\kappa_x\|_{p,I_{T^*}} \leq c_3\left(\|\kappa^0\|_{W_p^{2-2/p}(I)} + \|\rho^0\|_{W_p^{2-2/p}(I)} + 1\right),$$

where $c_3 = c_3(T^*) > 0$ is a positive constant which depends on T^*. Recall the special coupling of system (5.1); the brief introduction in the beginning of the proof of this proposition, and the above estimate, we can deduce that:

$$\|(\rho,\kappa)\|_{W_p^{2,1}(I_{T^*})} \leq c_4\left(\|\kappa^0\|_{W_p^{2-2/p}(I)} + \|\rho^0\|_{W_p^{2-2/p}(I)} + 1\right), \tag{5.16}$$

with $c_4 = c_4(T^*) > 0$ is also a positive constant depending on T^* but independent of the initial data.

Step 4. (The exponential estimate by iteration)

Now we move to show the exponential bound. Set

$$f(t) = \|(\rho,\kappa)\|_{W_p^{2,1}(I\times(t,t+T^*))}, \quad h(t) = \|(\rho_x,\kappa_x)\|_{\infty,I\times(t,t+T^*)},$$

and

$$g(t) = \|\kappa(\cdot,t)\|_{W_p^{2-2/p}(I)} + \|\rho(\cdot,t)\|_{W_p^{2-2/p}(I)}. \tag{5.17}$$

We have proved in Step 3, estimate (5.16), that

$$f(0) \leq c_4[g(0)+1],$$

and we know, from the Lemma 2.9 “trace of $W_p^{2,1}$ functions”, estimate (2.23), that

$$g(T^*) \leq c_5 f(0), \quad c_5 = c_5(T^*) > 0,$$

hence for $\lambda = 1 + c_4 c_5 > 1$, we get:

$$g(T^*) + 1 \leq \lambda[g(0) + 1].$$

Therefore, for $n \in \mathbb{N}$, $n \geq 1$, by iteration we have:

$$g(nT^*) + 1 \leq \lambda^n [g(0) + 1],$$

and hence

$$f(nT^*) \leq c_4 \lambda^n [g(0) + 1]. \tag{5.18}$$

From the Sobolev embedding in Hölder spaces, Lemma 2.8, estimate (2.19), we know that

$$h(nT^*) \leq c_6 f(nT^*), \quad c_6 = c_6(T^*) > 0. \tag{5.19}$$

Combining (5.18) and (5.19), we obtain

$$h(nT^*) \leq c_7 \lambda^n [g(0) + 1], \quad c_7 = c_4 c_6. \tag{5.20}$$

Using the fact that

$$h(t) \leq h(nT^*) + h((n+1)T^*), \quad \text{if} \quad nT^* \leq t \leq (n+1)T^*,$$

we deduce, from (5.20), that:

$$h(t) \leq c_8 [g(0) + 1] e^{c_9 t},$$

where

$$c_8 = (1 + \lambda) c_7, \quad c_9 = \frac{\mu}{T^*} \quad \text{with} \quad \mu = \log \lambda.$$

Since

$$\|(\rho_x(.,t), \kappa_x(.,t))\|_{\infty,I} \leq h(t),$$

the result easily follows. $\square$

Remark 5.3 ***(Exponential bound for $|\rho_x|^{(\alpha)}_{I\times(t,t+T^*)}$ and $|\kappa_x|^{(\alpha)}_{I\times(t,t+T^*)}$)***
We remark that from the Sobolev embedding in Hölder spaces (see Lemma 2.9):

$$W_p^{2,1}(I_T) \hookrightarrow C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{I_T}), \quad p > 3,$$

the previous result could be improved to an exponential bound of $|\rho_x|^{(\alpha)}_{I\times(t,t+T^*)}$ and $|\kappa_x|^{(\alpha)}_{I\times(t,t+T^*)}$, namely:

$$|\rho_x|^{(\alpha)}_{I\times(t,t+T^*)} \leq ce^{ct} \quad \textit{and} \quad |\kappa_x|^{(\alpha)}_{I\times(t,t+T^*)} \leq ce^{ct}, \tag{5.21}$$

where $c > 0$ is a positive constant only depending on the initial conditions.

Proposition 5.4 ***(Exponential bound in time for*** $\|\rho_{xx}(.,t)\|_{\infty,I}$***)***
Under the same hypothesis of Proposition 5.1, we have

$$\|\rho_{xx}(.,t)\|_{\infty,I} \leq cAe^{ct}, \quad t \geq 0, \tag{5.22}$$

where

$$A = 1 + \|\rho^0\|_{W_p^{2-2/p}(I)} + \|\kappa^0\|_{W_p^{2-2/p}(I)} + |\rho^0|_I^{(2+\alpha)},$$

and $c > 0$ *is a fixed positive constant independent of the initial data.*

Proof. Throughout the proof, we will omit, without loss of generality, the dependence on $\|B\|_\infty$. The ideas of the proof are somehow contained in the proof of the previous proposition. In fact, we will not only show the exponential bound for the L^∞ norm of ρ_{xx}, but also for the C^α norm. The proof is done in two steps.

Step 1. (Estimating ρ in the $C^{2+\alpha,\frac{2+\alpha}{2}}$ norm)

We start by writing down the Hölder estimate (2.11) for the second equation of (5.1). Indeed, since $\kappa_x \in C^{\alpha,\alpha/2}(\overline{I_T})$, and since the compatibility conditions of order 1 are satisfied, we have that:

$$|\rho|_{I_T}^{(2+\alpha)} \leq c(T)\left(|\kappa_x|_{I_T}^{(\alpha)} + |\rho^0|_I^{(2+\alpha)}\right). \tag{5.23}$$

We aim to control $|\kappa_x|_{I_T}^{(\alpha)}$ for an arbitrarily fixed small time. Following the same arguments of Steps 1 and 2 of Proposition 5.1, we get (for a sufficiently small time T) an estimate of $\|\bar{\kappa}\|_{W_p^{2,1}(I_T)}$, similar to (5.15), that reads:

$$\|\bar{\kappa}\|_{W_p^{2,1}(I_T)} \leq c(T)\left(1 + \|\rho^0\|_{W_p^{2-2/p}(I)} + \|\kappa^0\|_{W_p^{2-2/p}(I)}\right) + c\|\kappa_x\|_{p,I_T}, \tag{5.24}$$

where $\bar{\kappa}$ is given by (5.7). Using the Sobolev embedding in Hölder spaces, namely estimates (2.19) and (2.20), together with the fact that $\bar{\kappa} = 0$ on the parabolic boundary $\partial^p I_T$, we get:

$$\|\bar{\kappa}_x\|_{\infty,I_T} \leq c\left\{T^{\alpha/2}(\|\bar{\kappa}_t\|_{p,I_T} + \|\bar{\kappa}_{xx}\|_{p,I_T}) + T^{\frac{\alpha}{2}-1}\|\bar{\kappa}\|_{p,I_T}\right\} \leq cT^{\frac{p-3}{2p}}\|\bar{\kappa}\|_{W_p^{2,1}(I_T)}, \tag{5.25}$$

and

$$\langle\bar{\kappa}_x\rangle_{I_T}^{(\alpha)} \leq c\left\{\|\bar{\kappa}_t\|_{p,I_T} + \|\bar{\kappa}_{xx}\|_{p,I_T} + \frac{1}{T}\|\bar{\kappa}\|_{p,I_T}\right\} \leq c\|\bar{\kappa}\|_{W_p^{2,1}(I_T)}, \tag{5.26}$$

where p and α are always given by Remark 4.1. We notice that for the first equation (5.25), we have used Lemma 2.10 (the ideas are contained in the proof of this lemma, see Appendix A), while for the second one (5.26), we have applied estimate (2.17) for the term $\|\bar{\kappa}\|_{p,I_T}$. Combining (5.25) and (5.26), we deduce (for T small enough) that:

$$|\bar{\kappa}_x|_{I_T}^{(\alpha)} \leq c\|\bar{\kappa}\|_{W_p^{2,1}(I_T)}, \quad c > 0 \text{ independent of } T,$$

and hence, from (5.24) and the definition (5.7) of $\bar{\kappa}$, we obtain:

$$|\kappa_x|_{I_T}^{(\alpha)} \leq c(T)\left(1 + \|\rho^0\|_{W_p^{2-2/p}(I)} + \|\kappa^0\|_{W_p^{2-2/p}(I)}\right) + c\|\kappa_x\|_{p,I_T}. \tag{5.27}$$

For the term where it interferes the κ', we have used the following:

$$|\kappa'_x|_{I_T}^{(\alpha)} \le c(T)\|\kappa'\|_{W_p^{2,1}(I_T)} \le c(T)\left(\|\kappa^0\|_{W_p^{2-2/p}(I)} + 1\right).$$

Having in mind that the term $\|\kappa_x\|_{p,I_T}$ satisfies:

$$\|\kappa_x\|_{p,I_T} \le T^{1/p}|\kappa_x|_{I_T}^{(\alpha)},$$

inequality (5.27) can be written:

$$\left(1 - cT^{1/p}\right)|\kappa_x|_{I_T}^{(\alpha)} \le c(T)\left(1 + \|\rho^0\|_{W_p^{2-2/p}(I)} + \|\kappa^0\|_{W_p^{2-2/p}(I)}\right),$$

and hence for T^* small enough, namely

$$T^* = \frac{1}{2c^p},$$

we get

$$|\kappa_x|_{I_{T^*}}^{(\alpha)} \le c_{10}\left(1 + \|\rho^0\|_{W_p^{2-2/p}(I)} + \|\kappa^0\|_{W_p^{2-2/p}(I)}\right), \quad c_{10} = c_{10}(T^*) > 0. \tag{5.28}$$

Plugging (5.28) into (5.23), we deduce that:

$$|\rho|_{I_{T^*}}^{(2+\alpha)} \le c_{11}\left(1 + \|\rho^0\|_{W_p^{2-2/p}(I)} + \|\kappa^0\|_{W_p^{2-2/p}(I)} + |\rho^0|_I^{(2+\alpha)}\right), \ c_{11} = c_{11}(T^*) \ge 1. \tag{5.29}$$

Here we consider $c_{11} \ge 1$ for technical reasons.

Step 2. (The exponential estimate by iteration)

This is similar to Step 4 of Proposition 5.1. We first notice that the arguments presented in that step can be adapted to get an exponential bound on the function g given by (5.17). Indeed, we use (5.18) and the estimate of the traces of functions in Sobolev spaces (see Lemma 2.9, estimate (2.23)), to deduce that, for every $t \ge 0$:

$$g(t) \le c_{12}[1 + g(0)]e^{c_{12}t}, \tag{5.30}$$

with $c_{12} \ge 1$ is a fixed positive constant independent of the initial conditions. Also here $c_{11} \ge 1$ is taken for technical reasons. Let

$$\bar{f}(t) = |\rho|_{I\times(t,t+T^*)}^{(2+\alpha)}, \quad T^* \text{ is given in Step 1.}$$

From (5.29) and (5.30), we know that

$$\begin{aligned} \bar{f}(0) &\le c_{11}\left(1 + g(0) + |\rho^0|_I^{(2+\alpha)}\right) \\ &\le c_{11} + c_{11}c_{12}[1 + g(0)] + c_{11}|\rho^0|_I^{(2+\alpha)}. \end{aligned}$$

In a similar way, knowing that $c_{11} \ge 1$ and $c_{12} \ge 1$, we obtain:

$$\begin{aligned} \bar{f}(T^*) &\le c_{11}\left(1 + g(T^*) + \bar{f}(0)\right) \\ &\le 2c_{11}^2 + 2c_{11}^2 c_{12}[1 + g(0)]e^{c_{12}T^*} + c_{11}^2|\rho^0|_I^{(2+\alpha)}, \end{aligned}$$

and hence, by iteration, we get for every $n \in \mathbb{N}$:

$$\bar{f}(nT^*) \leq (n+1)c_{11}^{n+1} + (n+1)c_{11}^{n+1}c_{12}[1+g(0)]e^{nc_{12}T^*} + c_{11}^{n+1}|\rho^0|_I^{(2+\alpha)}.$$

From this inequality, and the fact that for $nT^* \leq t \leq (n+1)T^*$, we have $\bar{f}(t) \leq \bar{f}(nT^*) + \bar{f}((n+1)T^*)$, we easily arrive to the result (see the conclusion of Step 4 of Proposition 5.1). □

Remark 5.5 ***(Exponential bound for*** $|\rho|_{I\times(t,t+T^*)}^{(2+\alpha)}$***)***
Proposition 5.4, as it appears in the proof, gives an exponential bound, not only for $\|\rho(.,t)\|_{\infty,I}$*, but also for* $|\rho|_{I\times(t,t+T^*)}^{(2+\alpha)}$.

6 An upper bound for the $W_2^{2,1}$ norm of ρ_{xxx}

This section is devoted to give a suitable upper bound for the $W_2^{2,1}$ norm of ρ_{xxx}. This result will be a consequence of the control of the $W_2^{2,1}$ norm of κ_t and κ_{xx}. The goal is to use this upper bound in the Kozono-Taniuchi inequality (see inequality (2.33) of Theorem 2.16) in order to control the L^∞ norm of ρ_{xxx}. Let us consider the following hypothesis.

(H1). The term $\overline{T}$ is a fixed time that satisfies:

$$0 < T_1 \leq \overline{T}, \tag{6.1}$$

where T_1 is an arbitrarily small fixed number.

(H2). The function κ_x satisfies:

$$\kappa_x(x,t) \geq \gamma(t) > 0, \quad t \in [0,\overline{T}], \tag{6.2}$$

where $\gamma(t)$ is a positive decreasing function with $\gamma(0) < 1$.

Let

$$\mathcal{D} = I_{\overline{T}}, \tag{6.3}$$

we start with the first lemma.

Lemma 6.1 ***(***$W_2^{2,1}$ ***bound for*** κ_t ***and*** κ_{xx}***)***
Under hypothesis (H1)-(H2), and under the same hypothesis of Proposition 5.1, we have:

$$\|\kappa_t, \kappa_{xx}\|_{W_2^{2,1}(\mathcal{D})} \leq \frac{E}{\gamma^4},$$

where

$$\gamma := \gamma(\overline{T}),$$

and

$$E = de^{d\overline{T}},$$

with $d \geq 1$ *is a positive constant depending on the initial conditions but independent of* $\overline{T}$*, and will be given at the end of the proof.*

Remark 6.2 ***(The constant E depending on time)***
Let us stress on the fact that, througout the proof, the term $E = de^{d\overline{T}}$ of Lemma 6.1 might vary from line to line. In other words, the term d in the expression of E might certainly vary from line to line, but always satisfying the fact of just being dependent on the initial data of the problem. The different E's appearing in different estimates can be made the same by simply taking the maximum between them. Therefore they will all be denoted by the same letter E.

Proof. Define the functions u and v by:

$$u(x,t) = \rho_t(x,t) \quad \text{and} \quad v(x,t) = \kappa_t(x,t).$$

We write down the equations satisfied by u and v respectively:

$$\begin{cases} u_t = (1+\varepsilon)u_{xx} - \tau v_x & \text{on} \quad \mathcal{D}, \\ u|_{S_{\overline{T}}} = 0, \\ u|_{t=0} = u^0 := (1+\varepsilon)\rho^0_{xx} - \tau\kappa^0_x & \text{on} \quad I, \end{cases} \tag{6.4}$$

and with $B = \frac{\rho_x}{\kappa_x}$:

$$\begin{cases} v_t = \varepsilon v_{xx} + \dfrac{\rho_{xx}}{\kappa_x} u_x + B u_{xx} - B\dfrac{\rho_{xx}}{\kappa_x} v_x - \tau u_x & \text{on} \quad \mathcal{D}, \\ v|_{S_{\overline{T}}} = 0, \\ v|_{t=0} = v^0 := \varepsilon\kappa^0_{xx} + \dfrac{\rho^0_x\rho^0_{xx}}{\kappa^0_x} - \tau\rho^0_x & \text{on} \quad I. \end{cases} \tag{6.5}$$

The proof could be divided into three steps. As a first step, we will estimate the $L^\infty(\mathcal{D})$ norm of the term $v_x = \kappa_{tx}$. In the second step, we will control the $W^{2,1}_2(\mathcal{D})$ norm of $v = \kappa_t$. Finally, in the third step, we will show how to deduce a similar control on the $W^{2,1}_2(\mathcal{D})$ norm of κ_{xx}.

Step 1. (Estimating $\|v_x\|_{\infty,\mathcal{D}}$)

Since $v_x = \kappa_{tx}$, it is worth recalling the equation satisfied by κ:

$$\kappa_t = \varepsilon\kappa_{xx} + \frac{\rho_x\rho_{xx}}{\kappa_x} - \tau\rho_x. \tag{6.6}$$

In Step 3 of Proposition 4.6, we have shown that $\kappa \in C^{3+\alpha,\frac{3+\alpha}{2}}$. Therefore, writing the parabolic Hölder estimate (see (2.11)), we obtain:

$$\|\kappa_{tx}\|_{\infty,\mathcal{D}} \le |\kappa|^{(3+\alpha)}_{\mathcal{D}} \le c^H \left(1 + \left|\frac{\rho_x\rho_{xx}}{\kappa_x}\right|^{(1+\alpha)}_{\mathcal{D}} + |\rho_x|^{(1+\alpha)}_{\mathcal{D}}\right), \tag{6.7}$$

where the term 1 comes from the boundary conditions, and $c^H > 0$ is the positive constant given by (2.12) that can be estimated as $c^H \le E$. We use the elementary identity

$$|fg|^{(1+\alpha)}_{\mathcal{D}} \le \|f\|_{\infty,\mathcal{D}}|g|^{(1+\alpha)}_{\mathcal{D}} + \|g\|_{\infty,\mathcal{D}}|f|^{(1+\alpha)}_{\mathcal{D}} + \|f_x\|_{\infty,\mathcal{D}}|g|^{(\alpha)}_{\mathcal{D}} + \|g_x\|_{\infty,\mathcal{D}}|f|^{(\alpha)}_{\mathcal{D}},$$

to the term $\left|\frac{\rho_x\rho_{xx}}{\kappa_x}\right|_{\mathcal{D}}^{(1+\alpha)}$ with $f=\frac{\rho_x}{\kappa_x}$ and $g=\rho_{xx}$, we get:

$$\begin{aligned}\left|\frac{\rho_x\rho_{xx}}{\kappa_x}\right|_{\mathcal{D}}^{(1+\alpha)} &\leq 3|\rho|_{\mathcal{D}}^{(3+\alpha)}+\|\rho_{xx}\|_{\infty,\mathcal{D}}\left\langle\frac{\rho_x}{\kappa_x}\right\rangle_{\mathcal{D}}^{(1+\alpha)}+\|\rho_{xxx}\|_{\infty,\mathcal{D}}\left\langle\frac{\rho_x}{\kappa_x}\right\rangle_{\mathcal{D}}^{(\alpha)}\\ &+\frac{2|\rho|_{\mathcal{D}}^{(2+\alpha)}}{\gamma}(\|\rho_{xx}\|_{\infty,\mathcal{D}}+\|\kappa_{xx}\|_{\infty,\mathcal{D}}),\end{aligned}\tag{6.8}$$

where we have used the fact that $\kappa_x\geq\gamma$ and $\kappa_x\geq|\rho_x|$. We plug (6.8) in (6.7), we obtain:

$$\|\kappa_{tx}\|_{\infty,\mathcal{D}}\leq E\left(1+|\rho|_{\mathcal{D}}^{(3+\alpha)}+\left\langle\frac{\rho_x}{\kappa_x}\right\rangle_{\mathcal{D}}^{(1+\alpha)}+|\rho|_{\mathcal{D}}^{(3+\alpha)}\left\langle\frac{\rho_x}{\kappa_x}\right\rangle_{\mathcal{D}}^{(\alpha)}+\frac{1}{\gamma}\left(1+|\kappa|_{\mathcal{D}}^{(2+\alpha)}\right)\right),\tag{6.9}$$

where we have used used the fact that the term $|\rho|_{\mathcal{D}}^{(2+\alpha)}$ has an exponential bound (see Remark 5.5) of the form $|\rho|_{\mathcal{D}}^{(2+\alpha)}\leq E$. It is worth noticing that the term E appearing in (6.9) is the maximum between different E's that might exist as different bounds. This will be frequently used for the sake of simplicity.

Step 1.1. $\left(\textsf{Estimating}\left\langle\frac{\rho_x}{\kappa_x}\right\rangle_{\mathcal{D}}^{(1+\alpha)}\right)$

From the definition of the Hölder norm (see (2.4) and the notation therein), we see that in order to control $\left\langle\frac{\rho_x}{\kappa_x}\right\rangle_{\mathcal{D}}^{(1+\alpha)}$, it suffices to control the three quantities:

$$\left\langle\frac{\rho_x}{\kappa_x}\right\rangle_{t,\mathcal{D}}^{\left(\frac{1+\alpha}{2}\right)},\quad\left\langle\left(\frac{\rho_x}{\kappa_x}\right)_x\right\rangle_{x,\mathcal{D}}^{(\alpha)},\quad\text{and}\quad\left\langle\left(\frac{\rho_x}{\kappa_x}\right)_x\right\rangle_{t,\mathcal{D}}^{\left(\frac{\alpha}{2}\right)}.$$

We use the the following identity:

$$\left\langle\frac{f}{g}\right\rangle_{t,\mathcal{D}}^{(\alpha)}\leq\left\|\frac{f}{g}\right\|_{\infty,\mathcal{D}}\left\|\frac{1}{g}\right\|_{\infty,\mathcal{D}}\langle g\rangle_{t,\mathcal{D}}^{(\alpha)}+\left\|\frac{1}{g}\right\|_{\infty,\mathcal{D}}\langle f\rangle_{t,\mathcal{D}}^{(\alpha)},$$

with $f=\rho_x$ and $g=\kappa_x$, we get

$$\left\langle\frac{\rho_x}{\kappa_x}\right\rangle_{t,\mathcal{D}}^{\left(\frac{1+\alpha}{2}\right)}\leq\frac{1}{\gamma}\left(\langle\rho_x\rangle_{t,\mathcal{D}}^{\left(\frac{1+\alpha}{2}\right)}+\langle\kappa_x\rangle_{t,\mathcal{D}}^{\left(\frac{1+\alpha}{2}\right)}\right).\tag{6.10}$$

Similarly, we obtain:

$$\left\langle\frac{\rho_{xx}}{\kappa_x}\right\rangle_{x,\mathcal{D}}^{(\alpha)}\leq\frac{\|\rho_{xx}\|_{\infty,\mathcal{D}}}{\gamma^2}\langle\kappa_x\rangle_{x,\mathcal{D}}^{(\alpha)}+\frac{\langle\rho_{xx}\rangle_{x,\mathcal{D}}^{(\alpha)}}{\gamma}.\tag{6.11}$$

We also use the inequality:

$$\langle fg\rangle_{x,\mathcal{D}}^{(\alpha)}\leq\|f\|_{\infty,\mathcal{D}}\langle g\rangle_{x,\mathcal{D}}^{(\alpha)}+\|g\|_{\infty,\mathcal{D}}\langle f\rangle_{x,\mathcal{D}}^{(\alpha)},$$

with $f = \frac{\kappa_{xx}}{\kappa_x}$ and $g = \frac{\rho_x}{\kappa_x}$, we get:

$$\left\langle \frac{\kappa_{xx}\rho_x}{\kappa_x^2} \right\rangle_{x,\mathcal{D}}^{(\alpha)} \leq \frac{\langle \kappa_{xx} \rangle_{x,\mathcal{D}}^{(\alpha)}}{\gamma} + \frac{\|\kappa_{xx}\|_{\infty,\mathcal{D}}}{\gamma^2} \langle \rho_x \rangle_{x,\mathcal{D}}^{(\alpha)} + \frac{\|\kappa_{xx}\|_{\infty,\mathcal{D}}}{\gamma^2} \langle \kappa_x \rangle_{x,\mathcal{D}}^{(\alpha)}. \tag{6.12}$$

Similarly, we get

$$\left\langle \frac{\rho_{xx}}{\kappa_x} \right\rangle_{t,\mathcal{D}}^{\left(\frac{\alpha}{2}\right)} \leq \frac{\|\rho_{xx}\|_{\infty,\mathcal{D}}}{\gamma^2} \langle \kappa_x \rangle_{t,\mathcal{D}}^{\left(\frac{\alpha}{2}\right)} + \frac{\langle \rho_{xx} \rangle_{t,\mathcal{D}}^{\left(\frac{\alpha}{2}\right)}}{\gamma}, \tag{6.13}$$

and

$$\left\langle \frac{\kappa_{xx}\rho_x}{\kappa_x^2} \right\rangle_{t,\mathcal{D}}^{\left(\frac{\alpha}{2}\right)} \leq \frac{\langle \kappa_{xx} \rangle_{t,\mathcal{D}}^{\left(\frac{\alpha}{2}\right)}}{\gamma} + \frac{\|\kappa_{xx}\|_{\infty,\mathcal{D}}}{\gamma^2} \langle \rho_x \rangle_{t,\mathcal{D}}^{\left(\frac{\alpha}{2}\right)} + \frac{\|\kappa_{xx}\|_{\infty,\mathcal{D}}}{\gamma^2} \langle \kappa_x \rangle_{t,\mathcal{D}}^{\left(\frac{\alpha}{2}\right)}. \tag{6.14}$$

Collecting the above inequalities (6.10), (6.11), (6.12), (6.13), and (6.14) yield:

$$\left\langle \frac{\rho_x}{\kappa_x} \right\rangle_{\mathcal{D}}^{(1+\alpha)} \leq \frac{E}{\gamma^2} \left(1 + |\kappa|_{\mathcal{D}}^{(2+\alpha)} + \|\kappa_{xx}\|_{\infty,\mathcal{D}} \langle \kappa_x \rangle_{\mathcal{D}}^{(\alpha)} \right), \tag{6.15}$$

where we have used the fact that $1 \leq \frac{E}{\gamma}$, $\gamma \leq 1$ and $|\rho|_{\mathcal{D}}^{(2+\alpha)} \leq E$ (see Remark 5.5).

Step 1.2. $\Big($Estimating $|\rho|_{\mathcal{D}}^{(3+\alpha)}$ and $|\kappa|_{\mathcal{D}}^{(2+\alpha)}\Big)$

We recall the equation satisfied by ρ:

$$\rho_t = (1+\varepsilon)\rho_{xx} - \tau\kappa_x. \tag{6.16}$$

As for the term κ at the beginning of this Step 1, we have the following estimate for ρ:

$$|\rho|_{\mathcal{D}}^{(3+\alpha)} \leq E\left(1 + |\kappa|_{\mathcal{D}}^{(2+\alpha)}\right). \tag{6.17}$$

Having a second look at the equation (6.6) of κ, we can use again the parabolic Hölder estimate but for a lower order. In fact, we have:

$$|\kappa|_{\mathcal{D}}^{(2+\alpha)} \leq E\left(1 + \left|\frac{\rho_x\rho_{xx}}{\kappa_x}\right|_{\mathcal{D}}^{(\alpha)} + |\rho_x|_{\mathcal{D}}^{(\alpha)}\right).$$

Similar computations to those in Step 1.1 yield:

$$|\kappa|_{\mathcal{D}}^{(2+\alpha)} \leq \frac{E}{\gamma}\left(1 + |\kappa_x|_{\mathcal{D}}^{(\alpha)}\right), \tag{6.18}$$

and hence from (6.17), we also get a similar estimate for $|\rho|_{\mathcal{D}}^{(3+\alpha)}$:

$$|\rho|_{\mathcal{D}}^{(3+\alpha)} \leq \frac{E}{\gamma}\left(1 + |\kappa_x|_{\mathcal{D}}^{(\alpha)}\right). \tag{6.19}$$

Step 1.3. (The estimate for $\|\kappa_{tx}\|_{\infty,\mathcal{D}}$)

By combining (6.9), (6.15), (6.18) , (6.19), and by using the fact that $|\kappa_x|_{\mathcal{D}}^{(\alpha)}$ has an exponential estimate (see estimate (5.21) of Remark 5.3), we deduce that:

$$\|\kappa_{tx}\|_{\infty,\mathcal{D}} \leq \frac{E}{\gamma^3}, \tag{6.20}$$

where we have frequently used that $\gamma \leq 1$, and we have always taken the maximum of all the exponential bounds of the $E = de^{d\overline{T}}$ form.

Step 2. (Estimating $\|v\|_{W_2^{2,1}(\mathcal{D})}$)

We turn our attention to the equation (6.4) satisfied by u. We will show that we are in the good framework for applying the L^2 theory of parabolic equations. In fact, note first that $u = \rho_t \in C(\bar{\mathcal{D}})$, and hence the compatibility condition of order 0 is satisfied. Moreover, since $v_x = \kappa_{tx} \in C(\bar{\mathcal{D}})$ then $v_x \in L^2(\mathcal{D})$. Finally, the initial data satisfies $u^0 \in C^{1+\alpha}(\bar{I})$, hence $u^0 \in W_2^1(I)$. The above arguments show that the L^2 theory for parabolic equations (see Theorem 2.3) can be applied to the function u, therefore we get:

$$u \in W_2^{2,1}(\mathcal{D}) \implies \rho_t, \rho_{tt}, \rho_{tx}, \rho_{txx} \in L^2(\mathcal{D}),$$

with the following estimate:

$$\|u\|_{W_2^{2,1}(\mathcal{D})} \leq E(1 + \|v_x\|_{2,\mathcal{D}}). \tag{6.21}$$

Here the term E of the previous inequality hides in it all the constant c of the Sobolev estimate (see (2.16) and (2.17)), where this constant c behaves like $\overline{T}$ or $\sqrt{\overline{T}}$. Also the term 1 in (6.21) comes from the initial data. Since $v_x = \kappa_{tx}$, we plug the estimate (6.20) obtained in Step 1.3 into (6.21), we get

$$\|u\|_{W_2^{2,1}(\mathcal{D})} \leq E\left(1 + \sqrt{\overline{T}}\frac{E}{\gamma^3}\right).$$

Using some elementary identities, we finally obtain:

$$\|u\|_{W_2^{2,1}(\mathcal{D})} \leq \frac{E}{\gamma^3}. \tag{6.22}$$

Let us remind the reader that the term E is changing from line to line. We now consider equation (6.5) satisfied by v. In fact, for the same reasons as above with the new fact that $u \in W_2^{2,1}(\mathcal{D})$, we can easily deduce that we are in the good framework for the L^2 theory applied to v. Indeed, we have:

$$v \in W_2^{2,1}(\mathcal{D}) \implies \kappa_t, \kappa_{tt}, \kappa_{tx}, \kappa_{txx} \in L^2(\mathcal{D}),$$

with the following estimate:

$$\begin{aligned}\|v\|_{W_2^{2,1}(\mathcal{D})} \leq E\Bigg(1 + \left\|\tfrac{\rho_{xx}}{\kappa_x}\right\|_{\infty,\mathcal{D}} \|u_x\|_{2,\mathcal{D}} + \|B\|_{\infty,\mathcal{D}}\|u_{xx}\|_{2,\mathcal{D}} \\ + \|B\|_{\infty,\mathcal{D}} \left\|\tfrac{\rho_{xx}}{\kappa_x}\right\|_{\infty,\mathcal{D}} \|v_x\|_{2,\mathcal{D}} + \|u_x\|_{2,\mathcal{D}}\Bigg),\end{aligned} \tag{6.23}$$

hence from (6.20), (6.22), and some repeated computations, we deduce from (6.23) that:

$$\|\kappa_t\|_{W_2^{2,1}(\mathcal{D})} \leq \frac{E}{\gamma^4}. \tag{6.24}$$

As a byproduct of this last inequality, we can also get, using the Sobolev embedding Lemma (see Lemma 2.8-(ii)), that:

$$\|\kappa_t\|_{\infty,\mathcal{D}} \leq \frac{E}{\gamma^4}.$$

Remark that we can even get a better control by simply integrating (6.20) with respect to x, hence we obtain:

$$\|\kappa_t\|_{\infty,\mathcal{D}} \leq \frac{E}{\gamma^3}. \tag{6.25}$$

Step 3. (Estimating $\|\kappa_{xx}\|_{W_2^{2,1}(\mathcal{D})}$)

The estimate of $\|\kappa_{xx}\|_{W_2^{2,1}(\mathcal{D})}$ requires a special attention. We will mainly use the equations (6.16) and (6.6) satisfied by ρ and κ respectivly. The four parts $\|\kappa_{xx}\|_{2,\mathcal{D}}$, $\|\kappa_{xxt}\|_{2,\mathcal{D}}$, $\|\kappa_{xxx}\|_{2,\mathcal{D}}$ and $\|\kappa_{xxxx}\|_{2,\mathcal{D}}$ of the above norm will be estimated separately.

Step 3.1. (Estimate of $\|\kappa_{xx}\|_{2,\mathcal{D}}$)

This can be easily deduced from the equation (6.6) of κ. Indeed, this equation gives:

$$\begin{aligned} \|\kappa_{xx}\|_{2,\mathcal{D}} &\leq E\left(\|\kappa_t\|_{2,\mathcal{D}} + \sqrt{T}\|\rho_{xx}\|_{\infty,\mathcal{D}} + \sqrt{T}\|\rho_x\|_{\infty,\mathcal{D}}\right), \\ &\leq \frac{E}{\gamma^3}, \end{aligned} \tag{6.26}$$

where for the last line, we have used estimate (6.25), and the exponential bounds on $\|\rho_x\|_{\infty,\mathcal{D}}$ and $\|\rho_{xx}\|_{\infty,\mathcal{D}}$. Indeed, by the same way, we can even get, from the L^∞ bound (6.25) on κ_t, that

$$\|\kappa_{xx}\|_{\infty,\mathcal{D}} \leq \frac{E}{\gamma^3}. \tag{6.27}$$

Step 3.2. (Estimate of $\|\kappa_{xxt}\|_{2,\mathcal{D}}$)

As an immediate consequence of (6.24), we get

$$\|\kappa_{xxt}\|_{2,\mathcal{D}} \leq \frac{E}{\gamma^4}.$$

Step 3.3. (Estimate of $\|\kappa_{xxx}\|_{2,\mathcal{D}}$)

Using (6.19), we deduce that

$$\|\rho_{xxx}\|_{\infty,\mathcal{D}} \leq \frac{E}{\gamma}\left(1 + |\kappa_x|_{\mathcal{D}}^{(\alpha)}\right),$$

therefore, the fact that $|\kappa_x|_{\mathcal{D}}^{(\alpha)} \leq E$ (see Remark 5.3) gives:

$$\|\rho_{xxx}\|_{2,\mathcal{D}} \leq \frac{E}{\gamma}, \tag{6.28}$$

and hence (6.18) implies that:

$$\|\kappa_{xx}\|_{\infty,\mathcal{D}} \leq \frac{E}{\gamma}.$$

This will be used in estimating $\|\kappa_{xxt}\|_{2,\mathcal{D}}$. In fact, we derive the equation (6.6) satisfied by κ, with respect to x, we obtain:

$$\kappa_{tx} = \varepsilon\kappa_{xxx} + \frac{\rho_{xx}^2}{\kappa_x} + \frac{\rho_x\rho_{xxx}}{\kappa_x} - \frac{\rho_x\kappa_{xx}\rho_{xx}}{\kappa_x^2} - \tau\rho_{xx}, \tag{6.29}$$

and hence, using (6.28), we get:

$$\|\kappa_{xxx}\|_{2,\mathcal{D}} \leq \frac{E}{\gamma^3}. \tag{6.30}$$

Step 3.4. (Estimate of $\|\kappa_{xxxx}\|_{2,\mathcal{D}}$)

We first derive (6.16) two times in x, we deduce (using (6.22)) that $\|\rho_{xxxx}\|_{2,\mathcal{D}}$ has the same upper bound as $\|\kappa_{xxx}\|_{2,\mathcal{D}}$, i.e.

$$\|\rho_{xxxx}\|_{2,\mathcal{D}} \leq \frac{E}{\gamma^3}. \tag{6.31}$$

We derive the equation (6.29) once more with respect to x:

$$\begin{aligned}
\kappa_{txx} &= \varepsilon\kappa_{xxxx} + \frac{2\rho_{xx}\rho_{xxx}}{\kappa_x} - \frac{\kappa_{xx}\rho_{xx}^2}{\kappa_x^2} + \frac{\rho_x\rho_{xxxx}}{\kappa_x} - \frac{\rho_x\rho_{xxx}\kappa_{xx}}{\kappa_x^2} \\
&- \frac{\rho_{xx}^2\kappa_{xx}}{\kappa_x^2} - \frac{\rho_x\rho_{xx}\kappa_{xxx}}{\kappa_x^2} - \frac{\rho_x\kappa_{xx}\rho_{xxx}}{\kappa_x^2} + \frac{2\kappa_{xx}^2\rho_x\rho_{xx}}{\kappa_x^3} - \tau\rho_{xxx},
\end{aligned}$$

and we use (6.31) and our controls obtained in the previous steps, in order to deduce that:

$$\|\kappa_{xxxx}\|_{2,\mathcal{D}} \leq \frac{E}{\gamma^4}. \tag{6.32}$$

In fact, the highest power comes from estimating the following term:

$$\left\|\frac{\kappa_{xx}^2\rho_x\rho_{xx}}{\kappa_x^3}\right\|_{2,\mathcal{D}} \leq \left\|\frac{\kappa_{xx}^2\rho_{xx}}{\kappa_x^2}\right\|_{\infty,\mathcal{D}} \sqrt{T} \leq E\left(\frac{1}{\gamma}\right)^2\left(\frac{1}{\gamma^2}\right) = \frac{E}{\gamma^4},$$

where we have used the L^∞ estimate of $\|\kappa_{xx}\|_{\infty,\mathcal{D}}$. All other estimates are easily deduced. Let us just state how to estimate the other term were $\|\kappa_{xx}\|_{\infty,\mathcal{D}}$ interferes. In fact, we have:

$$\left\|\frac{\rho_x\rho_{xxx}\kappa_{xx}}{\kappa_x^2}\right\|_{2,\mathcal{D}} \leq \left\|\frac{\kappa_{xx}}{\kappa_x}\right\|_{\infty,\mathcal{D}} \|\rho_{xxx}\|_{2,\mathcal{D}} \leq E\left(\frac{1}{\gamma}\right)\left(\frac{1}{\gamma}\right)\left(\frac{1}{\gamma}\right).$$

Step 3.4. (Conclusion)

From the above estimates (6.26), (6.30) and (6.32), we finally deduce that:

$$\|\kappa_{xx}\|_{W_2^{2,1}(\mathcal{D})} \leq \frac{E}{\gamma^4}. \tag{6.33}$$

This terminates the proof. □

We move now to the main result of this section.

Lemma 6.3 ***($W_2^{2,1}$ bound for ρ_{xxx})***
Under the same hypothesis of Lemma 6.1, we have:

$$\|\rho_{xxx}\|_{W_2^{2,1}(\mathcal{D})} \leq \frac{E}{\gamma^4}.$$

Proof. From Step 3 of Proposition 4.6, we know that

$$(\rho, \kappa) \in C^\infty(\bar{I} \times [\delta, \overline{T}]), \quad \forall\, 0 < \delta < \overline{T}.$$

Therefore, we do the following computations over $\bar{\mathcal{D}} \setminus (\bar{I} \times \{t = 0\})$. Indeed, we derive twice the equation of ρ with respect to x, we get

$$\rho_{xxt} = (1+\varepsilon)\rho_{xxxx} - \tau\kappa_{xxx},$$

where on $S_{\overline{T}}$, we have:

$$(1+\varepsilon)\rho_{xx} = \tau\kappa_x \implies \rho_{xxt} = \frac{\tau\kappa_{xt}}{1+\varepsilon}.$$

Combining the above two equations, we obtain:

$$\rho_{xxxx} = \partial_x \left(\frac{\tau}{(1+\varepsilon)^2}\kappa_t + \frac{\tau}{1+\varepsilon}\kappa_{xx} \right) \quad \text{on} \quad S_{\overline{T}}. \tag{6.34}$$

Set

$$\bar{\kappa} = \frac{\tau}{(1+\varepsilon)^2}\kappa_t + \frac{\tau}{1+\varepsilon}\kappa_{xx} \quad \text{and} \quad w = \rho_{xxx}$$

and

$$\bar{w} = w - \bar{\kappa}.$$

We write down, after doing some computations, the equation satisfied by $\bar{w}$:

$$\begin{cases} \bar{w}_t = (1+\varepsilon)\bar{w}_{xx} - \dfrac{\tau}{(1+\varepsilon)^2}\kappa_{tt} \quad \text{on} \quad \mathcal{D} \\ \bar{w}_x|_{S_{\overline{T}}} = 0 \quad \text{on} \quad S_{\overline{T}} \\ \bar{w}|_{t=0} := \bar{w}^0 = \rho^0_{xxx} - \dfrac{\tau(1+2\varepsilon)}{(1+\varepsilon)^2}\kappa^0_{xx} - \dfrac{\tau}{(1+\varepsilon)^2}\dfrac{\rho^0_x\rho^0_{xx}}{\kappa^0_x} + \dfrac{\tau^2}{(1+\varepsilon)^2}\rho^0_x. \end{cases} \tag{6.35}$$

Let us show that the framework of the L^2 theory for parabolic equations with Neumann boundary conditions (see Theorem 2.3 and Remark 2.4 that follows) is well satisfied.

First, from Step 2 of Lemma 6.1, we know that $\kappa_{tt} \in L^2(\mathcal{D})$. Moreover, since we have supposed $(\rho^0, \kappa^0) \in (C^\infty(\bar{I}))^2$, then we eventually have $\bar{w}^0 \in W_2^1(I)$. We note that the compatibility conditions are not necessary in this case because the singular index in the Neumann framework is 3 (see Remark 2.4). These arguments permit to use the L^2 theory of parabolic equations with Neumann boundary conditions, hence we get:

$$\bar{w} \in W_2^{2,1}(\mathcal{D}),$$

and

$$\|\bar{w}\|_{W_2^{2,1}(\mathcal{D})} \leq E(1 + \|\kappa_{tt}\|_{2,\mathcal{D}}). \tag{6.36}$$

Since $\bar{w} = w - \bar{\kappa}$, we deduce, from (6.36), that:

$$\|\rho_{xxx}\|_{W_2^{2,1}(\mathcal{D})} \leq E\left(1 + \|\kappa_{tt}\|_{2,\mathcal{D}} + \|\kappa_t\|_{W_2^{2,1}(\mathcal{D})} + \|\kappa_{xx}\|_{W_2^{2,1}(\mathcal{D})}\right), \tag{6.37}$$

and eventually (6.37) with Lemma 6.1 gives immediately the result. $\square$

7 An upper bound for the BMO norm of ρ_{xxx}

This section is devoted to give a suitable upper bound for the BMO norm of ρ_{xxx}. This result will be a consequence of the control of the BMO norm of a suitable extension of κ_{xx}. As in the previous section, the goal is to use this upper bound in the Kozono-Taniuchi inequality (see inequality (2.33) of Theorem 2.16) in order to control the L^∞ norm of ρ_{xxx}. We first give some useful definitions.

Definition 7.1 ***(The "symmetric and periodic" extension of a function)***
Let $f \in C(\overline{I_T})$ be a continuous function, we define f^{sym} as the function constructed out of f, first by symmetry with respect to the line $x = 0$ over the interval $(-1, 0)$, i.e. $f(-x, t) = f(x, t)$, and then by spatial periodicity with $f(x + 2, t) = f(x, t)$.

Definition 7.2 ***(The "antisymmetric and periodic" extension of a function)***
Let $f \in C(\overline{I_T})$ be a continuous function, we define the function f^{asym} over $\mathbb{R} \times (0, \overline{T})$, first by the antisymmetry of f with respect to the line $x = 0$ over the interval $(-1, 0)$, i.e. $f(-x, t) = f(x, t)$, and then by spatial periodicity with $f(x + 2, t) = f(x, t)$.

We start with the following lemma that reflects a useful relation between the BMO norm of f^{sym} and f^{asym}.

Lemma 7.3 ***(A relation between f^{sym} and f^{asym})***
Let $f \in C(\overline{I_T})$, then:

$$\|f^{sym}\|_{BMO(\mathbb{R}\times(0,T))} \leq c\left(\|f^{asym}\|_{BMO(\mathbb{R}\times(0,T))} + m_{2I\times(0,T)}\left(|f^{sym}|\right)\right).$$

The proof of this lemma will be presented in Appendix B. The next lemma gives a control of the BMO norm of $(\kappa_{xx})^{asym}$.

Lemma 7.4 *(BMO **bound for** $(\kappa_{xx})^{asym}$)*
Under hypothesis (H1), and under the same hypothesis of Proposition 5.1, we have:

$$\|(\kappa_{xx})^{asym}\|_{BMO(\mathbb{R}\times(0,\overline{T}))} \leq ce^{c\overline{T}}, \tag{7.1}$$

where $c > 0$ is a constant depending on the initial conditions (but independent of $\overline{T}$). The function $(\kappa_{xx})^{asym}$ is given via Definition 7.2.

Proof. Let $\bar{\kappa}(x,t) = \kappa(x,t) - \kappa^0(x)$. We notice that $\bar{\kappa}|_{S_{\overline{T}}} = 0$, therefore $\bar{\kappa}^{asym}$ satisfies:

$$\begin{cases} \bar{\kappa}_t^{asym} = \varepsilon\bar{\kappa}_{xx}^{asym} + \dfrac{(\rho_x)^{asym}\rho_{xx}^{asym}}{(\kappa_x)^{asym}} - \tau(\rho_x)^{asym} + \varepsilon(\kappa_{xx}^0)^{asym} & \text{on} \quad \mathbb{R}\times(0,\overline{T}) \\ \bar{\kappa}^{asym}(x,0) = 0. \end{cases} \tag{7.2}$$

where, from Propositions 5.1 and 5.4, and the fact that

$$\left\|\frac{(\rho_x)^{asym}}{(\kappa_x)^{asym}}\right\|_{\infty,\mathbb{R}\times(0,\overline{T})} < 1,$$

we have:

$$\left\|\frac{(\rho_x)^{asym}\rho_{xx}^{asym}}{(\kappa_x)^{asym}} - \tau(\rho_x)^{asym} + \varepsilon(\kappa_{xx}^0)^{asym}\right\|_{\infty,\mathbb{R}\times(0,\overline{T})} \leq ce^{c\overline{T}}, \tag{7.3}$$

$c > 0$ is a constant depending on the initial conditions. From (7.3), we use the BMO theory for parabolic equations (Theorem 2.13), particularly (2.30), to deduce that:

$$\|\bar{\kappa}_{xx}^{asym}\|_{BMO(\mathbb{R}\times(0,\overline{T}))} \leq ce^{c\overline{T}},$$

and hence the result follows. □

We now present the principal result of this section.

Lemma 7.5 *(BMO **bound for** ρ_{xxx})*
Under hypothesis (H1)-(H2), and under the same hypothesis of Proposition 5.1, we have:

$$\|\rho_{xxx}\|_{BMO(\mathcal{D})} \leq E, \tag{7.4}$$

where E is the same as in Remark 6.2.

Proof. The proof is based on the following simple observation on the boundary $S_{\overline{T}}$. In fact, recall that the hölder regularity $C^{3+\alpha,\frac{3+\alpha}{2}}$, up to the boundary, for the solution (ρ,κ) permits using to conclude that:

$$\begin{cases} (1+\varepsilon)\rho_{xx} = \tau\kappa_x & \text{on} \quad \overline{S_{\overline{T}}} \\ (1+\varepsilon)\kappa_{xx} = \tau\rho_x & \text{on} \quad \overline{S_{\overline{T}}}. \end{cases}$$

hence a simple computation yields that:

$$\rho_{xx} = \partial_x\left(\frac{\tau\kappa}{1+\varepsilon}\right). \tag{7.5}$$

Let

$$\bar{\kappa} = \frac{\tau\kappa}{1+\varepsilon},$$

we write down the equation satisfied by $\bar{\kappa}$:

$$\begin{cases} \bar{\kappa}_t = \varepsilon\bar{\kappa}_{xx} + \dfrac{\tau}{1+\varepsilon}\dfrac{\rho_x\rho_{xx}}{\kappa_x} - \dfrac{\tau^2}{1+\varepsilon}\rho_x \quad \text{on} \quad \mathcal{D} \\ \bar{\kappa}|_{t=0} := \bar{\kappa}^0 = \dfrac{\tau\kappa^0}{1+\varepsilon} \quad \text{on} \quad I \\ \bar{\kappa}|_{S_{\overline{T}}} = \dfrac{\tau\kappa}{1+\varepsilon}\Big|_{S_{\overline{T}}} \\ \bar{\kappa}_x|_{S_{\overline{T}}} = \rho_{xx}. \end{cases} \tag{7.6}$$

Let

$$v = \rho_x,$$

we also write the equation satisfied by v:

$$\begin{cases} v_t = (1+\varepsilon)v_{xx} - \tau\kappa_{xx} \quad \text{on} \quad \mathcal{D} \\ v|_{t=0} = v^0 := \rho_x^0 \quad \text{on} \quad I \\ v|_{S_{\overline{T}}} = \rho_x \\ v_x|_{S_{\overline{T}}} = \rho_{xx}. \end{cases} \tag{7.7}$$

Take

$$\bar{v} = v - \bar{\kappa},$$

the equation satisfied by $\bar{v}$ reads:

$$\begin{cases} \bar{v}_t = (1+\varepsilon)\bar{v}_{xx} - \dfrac{\varepsilon\tau}{1+\varepsilon}\kappa_{xx} - \dfrac{\tau}{1+\varepsilon}\dfrac{\rho_x\rho_{xx}}{\kappa_x} + \dfrac{\tau^2}{1+\varepsilon}\rho_x \quad \text{on} \quad \mathcal{D} \\ \bar{v}|_{t=0} = \bar{v}^0 := \rho_x^0 - \dfrac{\tau}{1+\varepsilon}\kappa^0 \quad \text{on} \quad I \\ \bar{v}_x|_{S_{\overline{T}}} = 0. \end{cases} \tag{7.8}$$

We can assume, without loss of generality, that the initial condition $\bar{v}^0 = 0$. This is because being non-zero just adds a constant depending on the initial conditions in the final estimate that we are looking for. From the fact that $\bar{v}_x|_{S_{\overline{T}}} = 0$, we can easily deduce that the function $\bar{v}^{sym}$ satisfies:

$$\begin{cases} \bar{v}_t^{sym} = (1+\varepsilon)\bar{v}_{xx}^{sym} + \overbrace{\dfrac{\tau^2}{1+\varepsilon}(\rho_x)^{sym} - \dfrac{\tau}{1+\varepsilon}\dfrac{(\rho_x)^{sym}(\rho_{xx})^{sym}}{(\kappa_x)^{sym}} - \dfrac{\varepsilon\tau}{1+\varepsilon}(\kappa_{xx})^{sym}}^{g} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on} \quad \mathbb{R}\times(0,\overline{T}) \\ \bar{v}^{sym}(x,0) = 0 \quad \text{on} \quad \mathbb{R}, \end{cases}$$

therefore, using the BMO estimate (2.30) for parabolic equations, to the function $\bar{v}$, one gets:

$$\|\bar{v}_{xx}^{sym}\|_{BMO(\mathbb{R}\times(0,\overline{T}))} \le c\left[\|g\|_{BMO(\mathbb{R}\times(0,\overline{T}))} + m_{2I\times(0,\overline{T})}(|g|)\right]. \tag{7.9}$$

From Propositions 5.1, 5.4, we deduce that

$$\|g\|_{BMO(\mathbb{R}\times(0,\overline{T}))} \leq E + \|(\kappa_{xx})^{sym}\|_{BMO(\mathbb{R}\times(0,\overline{T}))}, \tag{7.10}$$

and

$$m_{2I\times(0,\overline{T})}(|g|) \leq E + m_{2I\times(0,\overline{T})}(|(\kappa_{xx})^{sym}|). \tag{7.11}$$

Recall the definition of the term E from Remark 6.2. At this stage, we write the following estimate:

$$\|(\kappa_{xx})^{sym}\|_{BMO(\mathbb{R}\times(0,\overline{T}))} \leq c\left[\|(\kappa_{xx})^{asym}\|_{BMO(\mathbb{R}\times(0,\overline{T}))} + m_{2I\times(0,\overline{T})}(|(\kappa_{xx})^{sym}|)\right], \tag{7.12}$$

which can be deduced using Lemma 7.3. The constant $c > 0$ appearing in (7.12) is independent of $\overline{T}$. Finally, we deduce that:

$$\begin{aligned}
\|\bar{v}_{xx}^{sym}\|_{BMO(\mathbb{R}\times(0,\overline{T}))} &\leq c\left[E + \|(\kappa_{xx})^{asym}\|_{BMO(\mathbb{R}\times(0,\overline{T}))} + m_{2I\times(0,\overline{T})}(|(\kappa_{xx})^{sym}|)\right]\\
&\leq c\left[E + m_{2I\times(0,\overline{T})}(|(\kappa_{xx})^{sym}|)\right]\\
&\leq c\left[E + (1/\overline{T})\|\kappa_{xx}\|_{1,\mathcal{D}}\right]\\
&\leq c\left[E + \overline{T}^{-1/p}\|\kappa_{xx}\|_{p,\mathcal{D}}\right],
\end{aligned}$$

where we have used (7.9), (7.10), (7.11) and (7.12) for the first line, and Lemma 7.4 for the second line. For the last line, we have used that $p > 3$. From (H1) and (5.16), we know that:

$$\overline{T}^{-1/p}\|\kappa_{xx}\|_{p,\mathcal{D}} \leq T_1^{-1/p}E.$$

From the above two inequalities, and since $\bar{v}_{xx} = \rho_{xxx} - \frac{\tau\kappa_{xx}}{1+\varepsilon}$, we easily arrive to our result. □

8 L^∞ bound for ρ_{xxx} and *revisited* results

In this section, we use the results of Sections 5, 6 and 7, in order to give an L^∞ bound for ρ_{xxx} via the Kozono-Taniuchi inequality. The next step is to improve some previously obtained results.

Proposition 8.1 *(L^∞ bound for ρ_{xxx})*
Under hypothesis (H1)-(H2), and under the same hypothesis of Proposition 5.1, we have:

$$\|\rho_{xxx}\|_{\infty,\mathcal{D}} \leq E\left(1 + \log^+ \frac{E}{\gamma^4}\right). \tag{8.1}$$

Proof. Applying estimate (2.33) to the function ρ_{xxx} over $\mathcal{D}$, we get:

$$\|\rho_{xxx}\|_{\infty,\mathcal{D}} \leq c\|\rho_{xxx}\|_{\overline{BMO}(\mathcal{D})}\left(1 + \log^+\|\rho_{xxx}\|_{W_2^{2,1}(\mathcal{D})} + \log^+\|\rho_{xxx}\|_{\overline{BMO}(\mathcal{D})}\right), \tag{8.2}$$

where we remind the reader that

$$\|\rho_{xxx}\|_{\overline{BMO}(\mathcal{D})} = \|\rho_{xxx}\|_{BMO(\mathcal{D})} + \|\rho_{xxx}\|_{1,\mathcal{D}}.$$

Using (8.2) together with Lemmas 7.5 and 6.3, lead to the result. The only term left to control is $\|\rho_{xxx}\|_{1,\mathcal{D}}$. In fact, we know that:

$$\|\rho_{xxx}\|_{1,\mathcal{D}} \leq \overline{T}^{1-\frac{1}{p}} \|\rho_{xxx}\|_{p,\mathcal{D}}, \tag{8.3}$$

and since, by repeating the same arguments of the proof of Lemma 7.5, and of Lemma 2.7 (see Appendix B), using the L^p estimates for parabolic equations instead of the BMO ones, we can conclude that:

$$\|\rho_{xxx}\|_{p,\mathcal{D}} \leq c(1 + \|\kappa_{xx}\|_{p,\mathcal{D}}),$$

where from (5.16), we finally get:

$$\|\rho_{xxx}\|_{p,\mathcal{D}} \leq ce^{c\overline{T}}.$$

This inequality together with (8.3) terminates the proof. $\square$

Remark 8.2 ***(Improving the comparison principle)***
The L^∞ bound on ρ_{xxx} given by Proposition 8.1 shows that we can improve our choice of the function γ of Proposition 3.1. Although the function γ was essentially used, on one hand, to ensure the positivity of κ_x for all time $t \geq 0$, and on the other hand, for the boundedness of the ratio $\frac{\rho_x}{\kappa_x}$, it was insufficient for showing the long time existence of (ρ, κ) given by Propositions 4.2 and 4.6; this lies from the fact that γ strongly depends, and in a dangerous way, on ρ_{xxx} (see inequality 3.30). The remedy of this inconvenience is to revisit the comparison principle "Proposition 3.1" with the new information given by Proposition 8.1, namely estimate (8.1).

Now, we show that we can even improve estimate (8.1) by eliminating the restrictive hypothesis (H1) and changing somehow the constant E appearing in (8.1). To be more precise, we write down our next corollary.

Corollary 8.3 ***(Proposition 8.1, revisited)***
Under hypothesis (H2), and under the same hypothesis of Proposition 5.1. Let

$$T > 0,$$

be any fixed time. Then we have:

$$\|\rho_{xxx}\|_{\infty,I_T} \leq E\left(1 + \log^+ \frac{E}{\gamma^4}\right). \tag{8.4}$$

Proof. We know, from Propositions 4.2, 4.6, used for $T_0 = 0$, that there exists some small $\delta_1 > 0$ only depending on the initial conditions, with:

$$\|\rho_{xxx}\|_{\infty,I_{\delta_1}} \leq c_{13}, \tag{8.5}$$

where $c_{13} > 0$ is a constant only depending on the initial conditions. We now apply Proposition 8.1 with

$$T_1 := \delta_1,$$

we get

$$\|\rho_{xxx}\|_{\infty,I_T} \le E\left(1+\log^+\frac{E}{\gamma^4}\right) \quad \text{if} \quad T \ge \delta_1, \tag{8.6}$$

where it is important to indicate that the term $E = E(\delta_1)$ appearing in (8.6) depends on $T_1 = \delta_1$ (see for instance the end of the proof of Lemma 7.5). Combining (8.5) and (8.6), we deduce that $\forall T > 0$:

$$\|\rho_{xxx}\|_{\infty,I_T} \le c_{13} + E\left(1+\log^+\frac{E}{\gamma^4}\right),$$

and hence the result follows. $\square$

The following proposition reflects how to improve Proposition 3.1.

Proposition 8.4 ***(The comparison principle, revisited)***
Under the same hypothesis of Corollary 8.3, and under the condition (3.2), we have:

$$\kappa_x(x,t) \ge \sqrt{\overline{\gamma}^2(t) + \rho_x^2(x,t)}, \quad \forall t \ge 0 \tag{8.7}$$

where $\overline{\gamma} > 0$ is a positive decreasing function depending on the initial conditions, and will be given in the proof.

Proof. In Proposition 3.1, we have that $\widetilde{c}$ (recall (3.1)) is a bound on $\|\rho_{xxx}\|_{\infty,I_T}$. From the *a priori* estimate (8.4) we can choose

$$\widetilde{c} = E(T)\left(1+\log^+\frac{E(T)}{\gamma^4(T)}\right), \tag{8.8}$$

for any $T > 0$. We assume that $\gamma(t)$ is a continuous decreasing function, and that the solution (ρ,κ) satisfies:

$$\kappa_x(x,t) \ge \gamma(t) > 0, \quad t \in [0,T].$$

Therefore, from the proof of Proposition 3.1, we have that

$$\overline{m} = \inf_{x\in I}\left(\cosh(\beta x)\left(\kappa_x - \sqrt{\gamma^2+\rho_x^2}\right)\right),$$

satisfies (3.26) on $(0,T)$. Here β satisfies (3.16) with $I = (-1,1)$. Therefore, using (8.8), we obtain

$$\overline{m}_t \ge -\left(\frac{E(T)\left(1+\log^+\frac{E(T)}{\gamma^4(T)}\right)}{\sqrt{\gamma^2+\rho_x^2}} + c_1\right)\overline{m} - \frac{c_2\gamma^2}{\sqrt{\gamma^2+\rho_x^2}} - \frac{\gamma\gamma'}{\sqrt{\gamma^2+\rho_x^2}}, \quad t \in (0,T), \tag{8.9}$$

with $c_1 = \frac{\beta^2}{4} + \frac{\tau^2}{8\varepsilon} + \varepsilon\beta^2$, and $c_2 = \frac{\tau^2\cosh\beta}{4\varepsilon}$. Since (8.9) is true for any $T > 0$, we deduce that:

$$\overline{m}_t(t) \ge -\left(\frac{E(t)\left(1+\log^+\frac{E(t)}{\gamma^4(t)}\right)}{\sqrt{\gamma^2(t)+\rho_x^2(x_0(t),t)}} + c_1\right)\overline{m}(t) - \frac{c_2\gamma^2(t)}{\sqrt{\gamma^2(t)+\rho_x^2(x_0(t),t)}}$$

$$-\frac{\gamma(t)\gamma^{'}(t)}{\sqrt{\gamma^2(t)+\rho_x^2(x_0(t),t)}}, \qquad t\in(0,T), \tag{8.10}$$

(recall the definition of x_0 by (3.27)). Following the same reasoning of the proof of Proposition 3.1, in particular Step 5, Case A, we know that, as long as $\overline{m}=\gamma^2$ is a solution of (8.10) with $\gamma\in C^1$, $\gamma^{'}<0$ (which is not the case in general), we have (see (3.30)):

$$\gamma^{'}(t)\geq-\left(c^*+E(t)\left(1+\log^+\frac{E(t)}{\gamma^4(t)}\right)\right)\gamma(t), \quad c^* \text{ given by (3.29)}, \ \ t\in(0,T). \tag{8.11}$$

Inequality (8.11) gives inspiration to the choice of $\overline{\gamma}$ as a solution of the following ODE:

$$\begin{cases}\overline{\gamma}^{'}(t)=-\left(c^*+E(t)\left(1+\log^+\dfrac{E(t)}{\overline{\gamma}^4(t)}\right)\right)\overline{\gamma}(t), \quad t\in(0,T)\\ \overline{\gamma}(0)=\alpha_2,\end{cases} \tag{8.12}$$

where α_2 is given by (3.33). It is easy to check that $\overline{\gamma}^2$ is a subsolution of (8.10), hence

$$\overline{m}\geq\overline{\gamma}^2>0,$$

with

$$\overline{m}(t)=\inf_{x\in I}\left(\cosh(\beta x)\left(\kappa_x(x,t)-\sqrt{\overline{\gamma}^2(t)+\rho_x^2(x,t)}\right)\right).$$

In this case, as long as,

$$\kappa_x(x,t)\geq\overline{\gamma}(t)>0, \quad \text{on} \quad [0,T], \tag{8.13}$$

we deduce that

$$\kappa_x(x,t)\geq\overline{\gamma}(t)+\frac{\overline{\gamma}^2(t)}{\cosh\beta}, \quad \text{on} \quad [0,T]. \tag{8.14}$$

Finally, from (8.13), (8.14) and the short-time existence result, Proposition 4.2, we easily deduce that $\kappa_x>0$ for all time and

$$\kappa_x(x,t)\geq\sqrt{\overline{\gamma}^2(t)+\rho_x^2(x,t)},$$

then the result follows. □

In fact, Proposition 8.4, can be used to improve our L^∞ exponential bounds found in Propositions 5.1 and 5.4. This will be the result of the next proposition.

Proposition 8.5 *Under the same hypothesis of Proposition 8.4. Let α_2 given by (3.23) satisfies:*

$$0<\alpha_2<1,$$

then the solution $(\rho,\kappa)\in C^{3+\alpha,\frac{3+\alpha}{2}}(\overline{I_T})$, $\forall T>0$, satisfies:

$$\kappa_x(.,t)\geq e^{-e^{e^{b(t+1)}}}, \quad \forall t\geq0, \tag{8.15}$$

$$|\rho(.,t)|_I^{(3+\alpha)} \le e^{e^{e^{b(t+1)}}}, \quad \forall t \ge 0, \tag{8.16}$$

and

$$|\kappa(.,t)|_I^{(3+\alpha)} \le e^{e^{e^{b(t+1)}}}, \quad \forall t \ge 0. \tag{8.17}$$

Here $b > 0$ is a positive constant depending on the initial conditions and the fixed terms of the problem, but independent of time.

Proof. The proof of this proposition could be divided into three steps.

Step 1. (Minoration of $\overline{\gamma}$ by $\underline{\gamma}$)

From the ODE (8.12) satisfied by $\overline{\gamma}$, and after doing some computations using the fact that $E(t) = de^{dt}$ is an increasing function over $(0,T)$, we get $\forall t \in (0,T)$:

$$\begin{aligned}
\overline{\gamma}'(t) &= -\left[c^* + E(t)\left(1 + \log^+ \frac{E(t)}{\gamma^4(t)}\right)\right]\overline{\gamma}(t) \\
&\ge -\left[c^* + E(T)\Big(1 + |\log d| + E(T) + 4|\log \overline{\gamma}(t)|\Big)\right]\overline{\gamma}(t) \\
&\ge -\underline{d}e^{\underline{d}T}\Big(1 + |\log \overline{\gamma}(t)|\Big)\overline{\gamma}(t),
\end{aligned}$$

where

$$\underline{d} = \max(4a, 2d), \quad \text{and} \quad a = \max\left(c^*, 4d, d|\log d|, d^2\right).$$

Let

$$\underline{E}(t) = \underline{d}e^{\underline{d}t}.$$

Define $\underline{\gamma}$ as the solution of the following ODE:

$$\begin{cases} \underline{\gamma}'(t) = -\underline{E}(T)\Big(1 + |\log \underline{\gamma}(t)|\Big)\underline{\gamma}(t), & t \in (0,T) \\ \underline{\gamma}(0) = \alpha_2. \end{cases} \tag{8.18}$$

From (8.18) and the above inequalities, we deduce that

$$\overline{\gamma}(t) \ge \underline{\gamma}(t), \quad \forall t \in (0,T).$$

Step 2. (Explicit minoration of $\overline{\gamma}$)

It is clear that the decreasing function

$$\underline{\gamma}_T(t) = e^{1-(1-\log \alpha_2)e^{(\underline{E}(T))t}} < 1 \tag{8.19}$$

is the solution of (8.18), and hence

$$\overline{\gamma}(t) \ge e^{1-(1-\log \alpha_2)e^{(\underline{E}(T))t}}, \quad t \in (0,T),$$

then we get (by the continuity of $\overline{\gamma}$ at $t = T$):

$$\begin{aligned}
\overline{\gamma}(t) &\ge e^{1-(1-\log \alpha_2)e^{\underline{d}te^{\underline{d}t}}} \\
&\ge e^{-e^{e^{b(t+1)}}}, \quad \forall t \ge 0,
\end{aligned} \tag{8.20}$$

for some constant $b > 0$ depending on the initial conditions and some other fixed terms, but independent of t. Inequality (8.15) directly follows from (8.7) and (8.20).

Step 3. (Estimate of $|\kappa|_{I_T}^{(3+\alpha)}$, $|\rho|_{I_T}^{(3+\alpha)}$ and conclusion)

From the proof of Lemma 6.1, we can easily deduce that the estimate of $\|\kappa_{tx}\|_{\infty,\mathcal{D}}$ (see (6.20)) is also true replacing $\|\kappa_{tx}\|_{\infty,\mathcal{D}}$ by $|\kappa|_{\mathcal{D}}^{(3+\alpha)}$. Therefore, from (6.19), (6.20) and (8.20), we deduce the result. $\square$

9 Long time existence and uniqueness

Now we are ready to show the main result of this paper, namely Theorem 1.1.

Proof of Theorem 1.1. Define the set $\mathcal{B}$ by:

$$\mathcal{B} = \left\{ \begin{array}{l} T > 0;\ \exists !\ \text{ solution } (\rho,\kappa) \in C^{3+\alpha,\frac{3+\alpha}{2}}(\overline{I_T}) \text{ of} \\ (1.1), (1.2) \text{ and } (1.3), \text{ satisfying } (1.11) \end{array} \right\}. \tag{9.1}$$

The proof could be divided into two steps.

Step 1. ($\mathcal{B}$ is a non-empty set)

The inequality (1.8) ensures the existence of $\overline{\gamma}(0) = \alpha_2 > 0$ given by (3.23), such that

$$\kappa_x^0 \geq \overline{\gamma}(0) \quad \text{on} \quad \bar{I}, \tag{9.2}$$

which together with (1.6) permits to apply the short-time existence result (Proposition 4.2). Hence there is some $T_1 > 0$ and a unique solution $(\rho,\kappa) \in W_p^{2,1}(I_{T_1})$, of (1.1), (1.2) and (1.3), with

$$\kappa_x \geq \frac{\overline{\gamma}(0)}{2} > 0 \quad \text{on} \quad \overline{I_{T_1}}. \tag{9.3}$$

From the boundary conditions of the initial data (1.7), we deduce, using Proposition 4.6, that this solution from $W_p^{2,1}(I_{T_1})$ is in fact $C^{3+\alpha,\frac{3+\alpha}{2}}(\overline{I_{T_1}})$ and therefore

$$|\rho_{xxx}| \leq \widetilde{c}_1 \quad \text{on} \quad \overline{I_{T_1}}, \tag{9.4}$$

for some $\widetilde{c}_1 > 0$. From (9.3), (9.4) and (1.8), we can use Proposition 3.1 where it follows that

$$\left|\frac{\rho_x}{\kappa_x}\right| \leq 1. \tag{9.5}$$

The above identities (9.3) and (9.5) show that

$$T_1 \in \mathcal{B},$$

and hence

$$\mathcal{B} \neq \emptyset.$$

Set
$$T_\infty = \sup \mathcal{B},$$
our next step is to prove that $T_\infty = \infty$.

Step 2. ($T_\infty = \infty$)

We will argue by contradiction. Suppose $T_1 \le T_\infty < \infty$. In this case, let $\delta > 0$ be an arbitrary small positive constant, then there exist some $T \in \mathcal{B}$ such that
$$T_\infty - \delta < T < T_\infty.$$
Since $T \in \mathcal{B}$, we recall from (8.15) that:
$$\kappa_x(.,t) \ge e^{-e^{e^{b(t+1)}}}, \quad \forall 0 \le t \le T,$$
and we recall from (8.16)-(8.17) that:
$$|\rho(.,t)|_I^{(2+\alpha)} \le e^{e^{e^{b(t+1)}}} \quad \text{and} \quad |\kappa(.,t)|_I^{(2+\alpha)} \le e^{e^{e^{b(t+1)}}}, \quad 0 \le t \le T. \tag{9.6}$$
We are going to apply Proposition 4.2 with $T_0 = T_\infty - \delta$. In fact, as a consequence of (8.15), we have:
$$\kappa_x(., T_\infty - \delta) \ge e^{-e^{e^{b(T_\infty - \delta + 1)}}} \ge e^{-e^{e^{b(T_\infty + 1)}}} =: \gamma_1 > 0. \tag{9.7}$$
Moreover, from (9.6), we deduce that
$$\|(D_x^s \kappa(., T_\infty - \delta), D_x^s \rho(., T_\infty - \delta))\|_{\infty, I} \le e^{e^{e^{b(T_\infty + 1)}}} =: M_1 \quad \text{for} \quad s = 1, 2. \tag{9.8}$$
From (9.7) and (9.8), we use Proposition 4.2 to obtain some
$$T^* = T^*(M_1, \gamma_1, \varepsilon, \tau, p) > 0 \tag{9.9}$$
such that there exists a unique solution $(\rho, \kappa) \in W_p^{2,1}(I \times (T_0, T_0 + T^*))$, $p = \frac{3}{1-\alpha}$, of (1.1), (4.3) and (4.4) with $T_0 = T_\infty - \delta$ and
$$\kappa_x \ge \frac{\gamma_1}{2} > 0 \quad \text{on} \quad \bar{I} \times [T_\infty - \delta, T_\infty - \delta + T^*]. \tag{9.10}$$
Again by (9.7) and (9.8), we can easily check that the quantities γ_1 and M_1 are independent of δ, and then T^* given by (9.9) is also independent of δ. However, we have by Proposition 8.4 that:
$$\kappa_x(., T_\infty - \delta) \ge \sqrt{\overline{\gamma}^2(T_\infty - \delta) + \rho_x^2(., T_\infty - \delta)},$$
then
$$\min_I \Big(\kappa_x(., T_\infty - \delta) - |\rho_x(., T_\infty - \delta)| \Big) > 0. \tag{9.11}$$
The compatibility conditions (4.41) and (4.42) are valid for $T_0 = T_\infty - \delta$ and this is due to the fact that the equation is satisfied in a strong sense up to the boundary where ρ

and κ are constants. This argument together with (9.11) permit, using first, Proposition 4.6 on the regularity $C^{3+\alpha,\frac{3+\alpha}{2}}$, and then Proposition 8.4 on the minoration of κ_x, to increase the regularity of this solution and then show that

$$\kappa_x > 0 \quad \text{and} \quad \left|\frac{\rho_x}{\kappa_x}\right| < 1 \quad \text{on} \quad \bar{I} \times [T_\infty - \delta, T_\infty - \delta + T^*]. \tag{9.12}$$

From (9.12) and the above arguments, we find that

$$T_\infty - \delta + T^* \in \mathcal{B},$$

with $T^* > 0$ independent of δ. By choosing

$$0 < \delta < T^*,$$

we deduce that

$$T_\infty - \delta + T^* > T_\infty,$$

which contradicts the definition of $T_\infty = \sup \mathcal{B}$. Therefore $T_\infty = \infty$. To complete the proof, we have to indicate that the C^∞ regularity (1.10) is automatically satisfied (see Step 3 of Proposition 4.6). □

10 Appendix A: miscellaneous parabolic estimates

A1. Proof of Lemma 2.7 (L^p estimate for parabolic equations)

As a first step, we will prove the result in the case where $\varepsilon = 1$, and in a second step, we will move to the case $\varepsilon > 0$. It is worth noticing that the term c may take several values only depending on p.

Step 1. (The estimate: case $\varepsilon = 1$)

Suppose $\varepsilon = 1$. Recall that $u \in W_p^{2,1}(I_T), p > 1$ is the unique solution of (2.1) with $f \in L^p(I_T)$ and $\phi = \Phi = 0$. Let $\tilde{u}$ be a special extension of the function u defined over $\mathbb{R} \times (0,T)$ by:

$$\begin{cases} \tilde{u}(x,t) = u(x,t) & \text{if} \quad 0 \le x \le 1 \\ \tilde{u}(x,t) = -u(2-x,t) & \text{if} \quad 1 \le x \le 2 \\ \tilde{u}(x+2,t) = \tilde{u}(x,t) & \text{otherwise.} \end{cases} \tag{10.1}$$

In exactly the same way, we can define $\tilde{f}$ out of the function f. It is easy to verify that $\tilde{u}$ satisfies:

$$\begin{cases} \tilde{u}_t = \tilde{u}_{xx} + \tilde{f} & \text{on} \quad \mathbb{R} \times (0,T) \\ \tilde{u}(x,0) = 0, & \text{on} \quad \mathbb{R}. \\ \tilde{u}(x,t) = 0, & x \in \mathbb{Z}. \end{cases} \tag{10.2}$$

Take a test function $\phi^n(x), n \in \mathbb{N}$ defined on $\mathbb{R}$ by:

$$\begin{cases} \phi^n(x) = 1 & \text{if} \;\; x \in (0, 2n) \\ \phi^n(x) = 0 & \text{if} \;\; x \ge 2n+1 \text{ or } x \le -1. \end{cases} \tag{10.3}$$

and set

$$J_T = 2I \times (0, T).$$

Define $\bar{u}$ by

$$\bar{u} = \tilde{u}\phi^n, \tag{10.4}$$

this function satisfies:

$$\begin{cases} \bar{u}_t = \bar{u}_{xx} + \bar{f}, & \text{on} \quad \mathbb{R} \times (0, T) \\ \bar{u}(x, 0) = 0, & \text{on} \quad \mathbb{R}, \end{cases} \tag{10.5}$$

with

$$\bar{f} = \tilde{f}\phi^n - \tilde{u}\phi^n_{xx} - 2\tilde{u}_x\phi^n_x. \tag{10.6}$$

The parabolic Calderon-Zygmund estimates (see [28, Proposition 7.11, page 168]) ensures the existence of a constant $c = c(p) > 0$ such that

$$\|\bar{u}_t\|_{p,\mathbb{R}\times(0,T)} + \|\bar{u}_{xx}\|_{p,\mathbb{R}\times(0,T)} \leq c\|\bar{f}\|_{p,\mathbb{R}\times(0,T)}, \tag{10.7}$$

where from (10.3), (10.4), (10.6) and (10.7), we deduce that

$$n(\|\tilde{u}_t\|_{p,J_T} + \|\tilde{u}_{xx}\|_{p,J_T}) + O(1) \leq cn\|\tilde{f}\|_{p,J_T} \tag{10.8}$$

with $O(1)$ remains bounded as $n \to \infty$. Dividing (10.8) by n and taking the limit as $n \to \infty$, we deduce that

$$\|\tilde{u}_t\|_{p,J_T} + \|\tilde{u}_{xx}\|_{p,J_T} \leq c\|\tilde{f}\|_{p,J_T},$$

hence by (10.1), we obtain

$$\|u_t\|_{p,I_T} + \|u_{xx}\|_{p,I_T} \leq c\|f\|_{p,I_T}, \quad c = c(p) > 0. \tag{10.9}$$

Since $u \in W^{2,1}_p(I_T)$ with $u|_{t=0} = 0$, we use [27, Lemma 4.5, page 305] to get

$$\|u\|_{p,I_T} \leq cT(\|u_t\|_{p,I_T} + \|u_{xx}\|_{p,I_T}) \tag{10.10}$$

and

$$\|u_x\|_{p,I_T} \leq c\sqrt{T}(\|u_t\|_{p,I_T} + \|u_{xx}\|_{p,I_T}). \tag{10.11}$$

Combining (10.9), (10.10) and (10.11), we deduce that

$$\frac{1}{T}\|u\|_{p,I_T} + \frac{1}{\sqrt{T}}\|u_x\|_{p,I_T} + \|u_{xx}\|_{p,I_T} + \|u_t\|_{p,I_T} \leq c\|f\|_{p,I_T}. \tag{10.12}$$

Step 2. (The estimate: general case $\varepsilon > 0$)

To get the general inequality, we consider the following rescaling of the function u:

$$\hat{u}(x, t) = u(x, t/\varepsilon), \quad (x, t) \in I_{\varepsilon T}, \tag{10.13}$$

which allows to get the desired result. □

A2. Proof of Lemma 2.10 (L^∞ control of the spatial derivative)

Since $u \in W_p^{2,1}(I_T)$ for $p > 3$, we know from Lemma 2.8 that $u_x \in C^{\alpha,\alpha/2}(\overline{I_T})$ for $\alpha = 1 - \frac{3}{p}$. In this case, we use the estimate (2.19) with $\delta = \sqrt{T}$, we obtain

$$\|u_x\|_{\infty,I_T} \leq c(p)\{T^{\frac{\alpha}{2}}(\|u_t\|_{p,I_T} + \|u_{xx}\|_{p,I_T}) + T^{\frac{\alpha}{2}-1}\|u\|_{p,I_T}\}. \tag{10.14}$$

Remark that the fact that $u = 0$ on the parabolic boundary $\partial^p I_T$, and that it satisfies the simple equation:

$$\begin{cases} u_t = u_{xx} + f, \quad f = u_t - u_{xx} \\ u = 0 \quad \text{on} \quad \partial^p I_T, \end{cases} \tag{10.15}$$

permits to apply estimate (2.17) to the function u. Hence (10.14) becomes (with a different nature of $c(p)$):

$$\begin{aligned} \|u_x\|_{\infty,I_T} &\leq c(p)\{T^{\frac{\alpha}{2}}\|u_t - u_{xx}\|_{p,I_T} + T^{\frac{\alpha}{2}-1}T\|u_t - u_{xx}\|_{p,I_T}\} \\ &\leq c(p)T^{\frac{\alpha}{2}}\|u\|_{W_p^{2,1}(I_T)} \\ &\leq c(p)T^{\frac{p-3}{2p}}\|u\|_{W_p^{2,1}(I_T)}, \end{aligned}$$

and the result follows. $\square$

11 Appendix B: parabolic BMO theory

B0. Proof of Lemma 7.3. We divide the proof into two steps.

Step 1. (treatment of small parabolic cubes)

Let us consider parabolic cubes $Q = Q_r(x_0, t_0) \subset \mathbb{R} \times (0, T)$ with $0 < r \leq \frac{1}{2}$. Assume, without loss of generality, that $1 < x_0 < 2$ (the other cases can be treated similarly). Define the left and the right neighbor cubes of $Q_r(x_0, t_0)$ by:

$$Q^- = Q_r^-(1 - r, t_0),$$

and

$$Q^+ = Q_r^+(1 + r, t_0),$$

respectively. Since $2r \leq 1$, then

$$Q^- \subset (0,1) \times (0,T) \quad \text{and} \quad Q^+ \subset (1,2) \times (0,T).$$

Using the fact that for any function $g \in L^1(\Omega)$:

$$\int_\Omega |g - m_\Omega(g)| \leq 2\int_\Omega |g - c|, \quad \forall c \in \mathbb{R},$$

We compute:

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |f^{sym}-m_Q(f^{sym})| &\leq \frac{2}{|Q|}\int_Q |f^{sym}+m_{Q^+}(f^{asym})| \\
&\leq \frac{2}{|Q^-|}\int_{Q^-} |f^{sym}+m_{Q^+}(f^{asym})| \\
&+ \frac{2}{|Q^+|}\int_{Q^+} |f^{sym}+m_{Q^+}(f^{asym})|. \qquad (11.1)
\end{aligned}$$

We know that from the properties of f^{sym} and f^{asym} that:

$$m_{Q^+}(f^{asym}) = -m_{Q^-}(f^{sym}),$$

and

$$f^{sym} = -f^{asym} \quad \text{on} \quad Q^+, \quad \text{and} \quad f^{sym} = f^{asym} \quad \text{on} \quad Q^-.$$

Using the above two inequalities in (11.1), we get:

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |f^{sym}-m_Q(f^{sym})| &\leq \frac{2}{|Q^-|}\int_{Q^-} |f^{sym}-m_{Q^-}(f^{sym})| \\
&+ \frac{2}{|Q^+|}\int_{Q^+} |f^{asym}-m_{Q^+}(f^{asym})| \\
&\leq \frac{2}{|Q^-|}\int_{Q^-} |f^{asym}-m_{Q^-}(f^{asym})| \\
&+ \frac{2}{|Q^+|}\int_{Q^+} |f^{asym}-m_{Q^+}(f^{asym})| \\
&\leq 4\|f^{asym}\|_{BMO(\mathbb{R}\times(0,T))}.
\end{aligned}$$

Step 2. (treatment of big parabolic cubes)

Consider parabolic cubes $Q = Q_r \subset \mathbb{R}\times(0,T)$ such that $r > \frac{1}{2}$. In this case, we compute:

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |f^{sym}-m_Q(f^{sym})| &\leq \frac{2}{|Q|}\int_Q |f^{sym}| \\
&\leq \frac{2N}{|Q|}\int_{2I\times(0,T)} |f^{sym}|,
\end{aligned}$$

with

$$|Q| \sim N \times |2I\times(0,T)|,$$

therefore

$$\frac{1}{|Q|}\int_Q |f^{sym}-m_Q(f^{sym})| \leq c\, m_{2I\times(0,T)}(|f^{sym}|).$$

Steps 1 and 2 directly implies the result. □

B1. Proof of Theorem 2.13 (BMO estimate for parabolic equations)

Let f be a bounded function defined on $\mathbb{R} \times (0,T)$ satisfying $f(x+2,t) = f(x,t)$. We extend the function f to $\mathbb{R} \times \mathbb{R}_+$, first by symmetry with respect to the line $\{t = T\}$ and after that by time periodicity of period $2T$; call this function $\tilde{f}$. Set $\bar{u}$ as the solution of the following equation:

$$\begin{cases} \bar{u}_t = \varepsilon \bar{u}_{xx} + \tilde{f} \quad \text{on} \quad \mathbb{R} \times \mathbb{R}_+ \\ \bar{u}(x,0) = 0. \end{cases} \tag{11.2}$$

We apply the standard result of BMO theory for parabolic equations. Since $f \in L^\infty(\mathbb{R} \times (0,T))$, then $\tilde{f} \in BMO(\mathbb{R} \times \mathbb{R}_+)$, and hence we obtain that

$$\bar{u}_t, \bar{u}_{xx} \in BMO(\mathbb{R} \times \mathbb{R}_+),$$

with the following estimate:

$$\|\bar{u}_t\|_{BMO(\mathbb{R}\times\mathbb{R}_+)} + \|\bar{u}_{xx}\|_{BMO(\mathbb{R}\times\mathbb{R}_+)} \leq c\|\tilde{f}\|_{BMO(\mathbb{R}\times\mathbb{R}_+)}, \tag{11.3}$$

and hence (from the definition of the BMO space),

$$\|u_t\|_{BMO(\mathbb{R}\times(0,T)} + \|u_{xx}\|_{BMO(\mathbb{R}\times(0,T))} \leq c\|\tilde{f}\|_{BMO(\mathbb{R}\times\mathbb{R}_+)}. \tag{11.4}$$

The BMO theory for parabolic equations, particularly estimate (11.3) is rather classical. This is due to the fact that the solution of (11.2) can be expressed in terms of the heat kernel Γ defined by:

$$\Gamma(x,t) = \begin{cases} (4\pi\varepsilon t)^{-1/2} e^{-\frac{x^2}{4\varepsilon t}}, & \text{for} \quad t > 0 \\ 0 & \text{for} \quad t \leq 0, \end{cases}$$

in the following way:

$$\bar{u}(x,t) = \int_{\mathbb{R}\times\mathbb{R}^+} \Gamma(x-\xi, t-s)\tilde{f}(\xi,s)\, d\xi\, ds.$$

As a matter of fact, it is shown in [19] that Γ_{xx} is a paraboic Calderon-Zygmund kernel (here we are working in nonhomogeneous metric spaces in which the variable t accounts for twice the variable x). Therefore $\Gamma_{xx} : BMO \to BMO$ is a bounded linear operator. This result is quite technical and can be adapted from its elliptic version (see [4, Theorem 3.4]). It is less difficult to show that $\Gamma_{xx} : L^\infty \to BMO$, a bounded linear operator (see for instance [24, Lemma 3.3]).

Having (11.4) in hands, it remains to show that

$$\|\tilde{f}\|_{BMO(\mathbb{R}\times\mathbb{R}_+)} \leq c\left(\|f\|_{BMO(\mathbb{R}\times(0,T))} + m_{2I\times(0,T)}(|f|)\right), \tag{11.5}$$

with $c > 0$ independent of T. This can be divided into three steps:

Step 1. (treatment of small parabolic cubes)

We consider parabolic cubes $Q_r(x_0,t_0)$, $(x_0,t_0) \in \mathbb{R} \times \mathbb{R}_+$, with

$$r \leq \sqrt{T}.$$

Let us estimate the term

$$\frac{1}{|Q_r|}\int_{Q_r}|\tilde{f}-m_{Q_r}\tilde{f}|.$$

Assume, without loss of generality, that

$$T \le t_0 < 2T.$$

In fact, any other case can be done in a similar way because of the time symmetry of the function $\tilde{f}$. Define Q_r^a and Q_r^b; the above and the below parabolic cubes, as follows:

$$Q_r^a = Q_r(x_0, T+r^2) \quad \text{and} \quad Q_r^b = Q_r(x_0, T).$$

Clearly

$$Q_r \subset (Q_r^a \cup Q_r^b) \quad \text{and} \quad |Q_r| = |Q_r^a| = |Q_r^b|.$$

We compute:

$$\begin{aligned}
\frac{1}{|Q_r|}\int_{Q_r}|\tilde{f}-m_{Q_r}\tilde{f}| &\le \frac{2}{|Q_r|}\int_{Q_r}|\tilde{f}-2m_{Q_r^b}\tilde{f}+m_{Q_r^a}\tilde{f}| \\
&\le \frac{4}{|Q_r|}\int_{Q_r}|\tilde{f}-m_{Q_r^b}\tilde{f}| + \frac{2}{|Q_r|}\int_{Q_r}|\tilde{f}-m_{Q_r^a}\tilde{f}| \\
&\le \frac{4}{|Q_r|}\int_{Q_r^a}|\tilde{f}-m_{Q_r^b}\tilde{f}| + \frac{4}{|Q_r|}\int_{Q_r^b}|\tilde{f}-m_{Q_r^b}\tilde{f}| \\
&\quad + \frac{2}{|Q_r|}\int_{Q_r^a}|\tilde{f}-m_{Q_r^a}\tilde{f}| + \frac{2}{|Q_r|}\int_{Q_r^b}|\tilde{f}-m_{Q_r^a}\tilde{f}|.
\end{aligned}$$

We remark (from the symmetry-in-time of the function $\tilde{f}$) that:

$$m_{Q_r^a}\tilde{f} = m_{Q_r^b}f,$$

and

$$\int_{Q_r^a}|\tilde{f}-c| = \int_{Q_r^b}|f-c|, \quad \forall c \in \mathbb{R}.$$

Therefore the above inequalities give that:

$$\frac{1}{|Q_r|}\int_{Q_r}|\tilde{f}-m_{Q_r}\tilde{f}| \le 16\|f\|_{BMO(\mathbb{R}\times\mathbb{R}_+)}. \tag{11.6}$$

Step 2. (treatment of big parabolic cubes)

Consider now parabolic cubes $Q_r \subset \mathbb{R}\times\mathbb{R}_+$, $r > \sqrt{T}$. Because of the symmetry-in-time of the function $\tilde{f}$, and its spatial periodicity, we compute:

$$\begin{aligned}
\frac{1}{|Q_r|}\int_{Q_r}|\tilde{f}-m_{Q_r}\tilde{f}| &\le \frac{2}{|Q_r|}\int_{Q_r}|\tilde{f}| \\
&\le \frac{2N}{|Q_r|}\int_{2I\times(0,T)}|f|,
\end{aligned}$$

Where

$$|Q_r| \sim N \times |2I \times (0,T)|.$$

Therefore, the above inequalities give:

$$\frac{1}{|Q_r|} \int_{Q_r} |\tilde{f} - m_{Q_r} \tilde{f}| \leq c\, m_{2I \times (0,T)}(|f|). \tag{11.7}$$

Step 3. (conclusion)

Combining (11.6) and (11.7), we obtain our result. □

B2. Sketch of the proof of Theorem 2.16 (a Kozono-Taniuchi parabolic type inequality)

The proof of the Kozono-Taniuchi parabolic type inequality will be a consequence of the following theorem where we give an analogue estimate over $\mathbb{R}_x \times \mathbb{R}_t$. More precisely, we have:

Theorem 11.1 ***(A Kozono-Taniuchi space-time parabolic type inequality)*** *Let $u \in C_0^\infty(\mathbb{R}^2)$, supp $u \in Q_R$. Then we have*

$$\|u\|_{\infty,\mathbb{R}^2} \leq c\|u\|_{\overline{BMO}(\mathbb{R}^2)} \left(1 + \log^+ \|u\|_{\overline{BMO}(\mathbb{R}^2)} + \log^+ \|u\|_{W_2^{2,1}(\mathbb{R}^2)}\right), \tag{11.8}$$

where

$$\|u\|_{\overline{BMO}} = \|u\|_{BMO} + \|u\|_{L^1},$$

and $c = c(R) > 0$ is a positive constant.

Sketch of the Proof of Theorem 11.1. First, we need to define some notations and spaces. Let $X = (x,t) \in \mathbb{R}^2$, we define the parabolic distance of X from the origin by:

$$|X|_p = (x^4 + t^2)^{1/2} \sim x^2 + |t|. \tag{11.9}$$

We write the parabolic version of the Littlewood-Paley dyadic decomposition. Let $\phi_j(X)$ be the inverse Fourier transform of the j-th component of the parabolic dyadic decomposition $\hat{\phi} = \{\hat{\phi}_j(\xi)\}_{j=0}^\infty \subset S(\mathbb{R}^2)$, $S(\mathbb{R}^2)$ is the Schwartz space, with

$$\begin{aligned} &\text{supp } \hat{\phi}_0 \subset \{\xi;\ |\xi|_p \leq 2\}, \\ &\text{supp } \hat{\phi}_j \subset \{\xi;\ 2^{j-1} \leq |\xi|_p \leq 2^{j+1}\} \quad \text{if} \quad j \in \mathbb{N}, \quad j \geq 1. \end{aligned} \tag{11.10}$$

Here $\xi = (\xi_x, \xi_t) \in \mathbb{R}^2$, and we have:

$$\sum_0^\infty \hat{\phi}_j(\xi) = 1.$$

The Lizorkin-Triebel space $\dot{F}^\gamma_{p,\rho}$

Let $\gamma \geq 0$. Let $1 \leq p < \infty$, $1 \leq \rho \leq \infty$ (or $p = \infty$, $1 \leq \rho < \infty$). We define the parabolic Lizorkin-Triebel space by

$$\dot{F}^{\gamma}_{p,\rho} = \{u \in S^{'}(\mathbb{R}^2);\ \|u\|_{\dot{F}^{\gamma}_{p,\rho}} < \infty\}, \tag{11.11}$$

where

$$\|u\|_{\dot{F}^{\gamma}_{p,\rho}} = \left\| \left(\sum_{j=0}^{\infty} 2^{j\gamma\rho} |\phi_j * u|^{\rho} \right)^{1/\rho} \right\|_{p,\mathbb{R}^2}. \tag{11.12}$$

The ideas of the proof could be separated into several steps.

Step 1. Let $\gamma > 0$. We compute:

$$\begin{aligned}
\|u\|_{\infty} &\leq \|u\|_{\dot{F}^0_{\infty,1}} \\
&\leq \left\| \sum_{0\leq j\leq N} |\phi_j * u| \right\|_{\infty} + \left\| \sum_{j>N} 2^{-j\gamma} 2^{j\gamma} |\phi_j * u| \right\|_{\infty} \\
&\leq \left\| N^{1/2} \left(\sum_{0\leq j<N} |\phi_j * u|^2 \right)^{1/2} \right\|_{\infty} + c_{\gamma} 2^{-\gamma N} \left\| \left(\sum_{j\geq N} \left(2^{j\gamma}|\phi_j * u|\right)^2 \right)^{1/2} \right\|_{\infty} \\
&\leq N^{1/2} \|u\|_{\dot{F}^0_{\infty,2}} + c_{\gamma} 2^{-\gamma N} \|u\|_{\dot{F}^{\gamma}_{\infty,2}},
\end{aligned} \tag{11.13}$$

where $c_{\gamma} \simeq \frac{1}{\gamma}$. Now we optimize (11.13) in N. For each u, we set

$$N \simeq \log_{2^{\gamma}} \left(c_{\gamma} \frac{\|u\|_{\dot{F}^{\gamma}_{\infty,2}}}{\|u\|_{\dot{F}^0_{\infty,2}}} \right),$$

we finally obtain

$$\|u\|_{\infty} \leq \|u\|_{\dot{F}^0_{\infty,1}} \leq c \|u\|_{\dot{F}^0_{\infty,2}} \left(1 + \left(\frac{1}{\gamma} \log^{+} \frac{\|u\|_{\dot{F}^{\gamma}_{\infty,2}}}{\|u\|_{\dot{F}^0_{\infty,2}}} \right)^{1/2} \right). \tag{11.14}$$

Step 2. Using the fact that $u \in C_0^{\infty}(\mathbb{R}^2)$, we get:

$$|\phi_0 * u| \leq c\|u\|_{L^1}$$

and

$$|\phi_j * u| \leq c\|u\|_{BMO}, \quad \forall j \geq 1,$$

and then we obtain:

$$\|u\|_{\dot{F}^0_{\infty,2}} \leq \|u\|^{1/2}_{\overline{BMO}} \|u\|^{1/2}_{\dot{F}^0_{\infty,1}}. \tag{11.15}$$

Using (11.14) with (11.15), we deduce that:

$$\frac{\|u\|_{\dot{F}^0_{\infty,1}}}{\|u\|_{\overline{BMO}}} \leq c \left(1 + \log^{+} \|u\|_{\dot{F}^{\gamma}_{\infty,2}} + \log^{+} \|u\|_{\overline{BMO}} + \log^{+} \frac{\|u\|_{\dot{F}^0_{\infty,1}}}{\|u\|_{\overline{BMO}}} \right),$$

hence

$$\|u\|_\infty \le \|u\|_{\dot F^0_{\infty,1}} \le c\|u\|_{\overline{BMO}} \left(1 + \log^+ \|u\|_{\dot F^\gamma_{\infty,2}} + \log^+ \|u\|_{\overline{BMO}}\right). \tag{11.16}$$

Step 3. ($\|u\|_{\dot F^\gamma_{\infty,2}} \le c\|u\|_{W^{2,1}_2}$, with $0<\gamma<\frac{1}{2}$)

Recall that

$$\|u\|_{\dot F^\gamma_{\infty,2}} = \left\| \left(\sum_{j\ge 0} \left(2^{\gamma j}|\phi_j * u|\right)^2 \right)^{1/2} \right\|_\infty .$$

We calculate

$$\begin{aligned} 2^{\gamma j}(\phi_j * u)(0) &= 2^{\gamma j} \int \hat\phi_j^* \cdot \hat{\check u} \\ &= 2^{\gamma j} \int \frac{\hat\phi_j^*}{\xi_x^2 + |\xi_t|} \cdot \hat{\check u} \cdot (\xi_x^2 + |\xi_t|), \end{aligned}$$

where ϕ_j^* is the complex conjugate of ϕ_j, and $\check u(x) = u(-x)$. Therefore, from Cauchy-Schwartz inequality and the fact that

$$\hat\phi_j = 0 \quad \text{if} \quad (\xi_x^2 + |\xi_t|)^{1/2} < 2^{j-1},$$

we obtain:

$$\begin{aligned} 2^{\gamma j}|\phi_j * u| &\le 2^{\gamma j} \left(\int \frac{\hat\phi_j^2}{(\xi_x^2 + |\xi_t|)^2} \right)^{1/2} \left(\int |\hat u|^2 (\xi_x^2 + |\xi_t|)^2 \right)^{1/2} \\ &\le \frac{c}{2^{j(\frac{1}{2}-\gamma)}} \|u\|_{W^{2,1}_2}. \end{aligned}$$

Finally, we get

$$\|u\|_{\dot F^\gamma_{\infty,2}} \le c\|u\|_{W^{2,1}_2} \left(\sum_{j\ge 0} \frac{1}{2^{2j(\frac{1}{2}-\gamma)}} \right)^{1/2}, \tag{11.17}$$

where the above series converges since $\gamma < \frac{1}{2}$.

Step 4. (Conclusion)

Combining (11.16) form Step 2, and (11.17) from Step 3, we get the required result. □

Back to the sketch of the proof of Theorem 2.16. Let v defined on $I \times (0,T)$. Take $\tilde v$ as the special extension of the function v defined as follows:

$$\tilde v(x,t) = -3v(-x,t) + 4v(-x/2,t) \quad \forall -1 < x < 0.$$

The continuation to $\mathbb{R} \times (0,T)$ is made by spatial periodicity. The extension in time will be done, first by symmetry with respect to $\{t=0\}$, and after that by time periodicity of period $2T$. Define the two zones $\mathcal{Z}_1$ and $\mathcal{Z}_2$ as follows:

$$\mathcal{Z}_1 = \{(x,t);\ -1/3 < x < 4/3,\ -T/3 < t < 4T/3\},$$

and

$$\mathcal{Z}_2 = \{(x,t);\ -2/3 < x < 5/3,\ -2T/3 < t < 5T/3\}.$$

Take ψ a cut-off function such that

$$\psi = 1 \text{ on } \mathcal{Z}_1, \quad \text{and} \quad \psi = 0 \text{ on } \mathbb{R}^2 \setminus \mathcal{Z}_2.$$

Let

$$u = \tilde{v}\psi,$$

we apply Theorem 11.1 to the function u, we get

$$\|v\|_{\infty,I_T} \le c\|\tilde{v}\psi\|_{\overline{BMO}(\mathbb{R}^2)} \left(1 + \log^+ \|\tilde{v}\psi\|_{\overline{BMO}(\mathbb{R}^2)} + \log^+ \|\tilde{v}\psi\|_{W_2^{2,1}(\mathbb{R}^2)}\right). \tag{11.18}$$

The special extension of the function v permits to write:

$$\|\tilde{v}\psi\|_{W_2^{2,1}(\mathbb{R}^2)} \le c\|v\|_{W_1^{2,1}(I_T)}. \tag{11.19}$$

Moreover, repeating similar arguments as in the proof of Theorem 2.13, Steps 1 and 2, we can treat relatively small cubes Q^s or relatively big cubes Q^b for the BMO norm of $\|\tilde{v}\psi\|_{\overline{BMO}(\mathbb{R}^2)}$. As a final consequence, we get

$$\frac{1}{|Q^i|}\int |\tilde{v}\psi - m_{Q^i}(\tilde{v}\psi)| \le \|v\|_{\overline{BMO}(I_T)}, \quad i \in \{s,b\}.$$

The only new case that we need to take care about is when the cube intersects the zone $\mathcal{Z}_2 \setminus \mathcal{Z}_1$ where $\psi \ne 0, 1$. In this case we use the fact that

$$\|\tilde{v}\psi\|_{BMO} \le c\|v\|_{BMO} + \|\tilde{v}\psi\|_{L^1},$$

which return us to one of the above two cases considered above. Therefore, we obtain

$$\|\tilde{v}\psi\|_{\overline{BMO}(\mathbb{R}^2)} \le c\|v\|_{\overline{BMO}(I_T)}. \tag{11.20}$$

From (11.18), (11.19) and (11.20), the result follows. □

References

[1] R. A. Adams, *Sobolev spaces*, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Pure and Applied Mathematics, Vol. 65.

[2] V. Alexiades, *A singular parabolic initial-boundary value problem in a noncylindrical domain*, SIAM J. Math. Anal., 11 (1980), pp. 348–357.

[3] J. T. Beale, T. Kato, and A. Majda, *Remarks on the breakdown of smooth solutions for the 3-D Euler equations*, Comm. Math. Phys., 94 (1984), pp. 61–66.

[4] M. Bramanti and L. Brandolini, *Estimates of BMO type for singular integrals on spaces of homogeneous type and applications to hypoelliptic PDEs*, Rev. Mat. Iberoamericana, 21 (2005), pp. 511–556.

[5] H. Brézis and T. Gallouët, *Nonlinear Schrödinger evolution equations*, Nonlinear Anal., 4 (1980), pp. 677–681.

[6] H. Brézis and S. Wainger, *A note on limiting cases of Sobolev embeddings and convolution inequalities*, Comm. Partial Differential Equations, 5 (1980), pp. 773–789.

[7] C. Y. Chan and H. G. Kaper, *Quenching for semilinear singular parabolic problems*, SIAM J. Math. Anal., 20 (1989), pp. 558–566.

[8] C. Y. Chan and C. C. S., *A numerical method for semilinear singular parabolic quenching problems*, Quart. Appl. Math., 47 (1989), pp. 45–57.

[9] C. Y. Chan and B. M. Wong, *Existence of classical solutions for singular parabolic problems*, Quart. Appl. Math., 53 (1995), pp. 201–213.

[10] Y. Z. Chen and E. DiBenedetto, *On the Harnack inequality for nonnegative solutions of singular parabolic equations*, in Degenerate diffusions (Minneapolis, MN, 1991), vol. 47 of IMA Vol. Math. Appl., Springer, New York, 1993, pp. 61–69.

[11] D. Colton, *Cauchy's problem for a singular parabolic partial differential equation*, J. Differential Equations, 8 (1970), pp. 250–257.

[12] E. DiBenedetto, *Degenerate parabolic equations*, Universitext, Springer-Verlag, New York, 1993.

[13] ——, *Degenerate and singular parabolic equations*, Recent advances in partial differential equations, RAM Res. Appl. Math., 30 (1994), pp. 55–84.

[14] E. DiBenedetto and Y.-Z. Chen, *On the local behaviour of solutions of singular parabolic equations*, J. Rational Mech. Anal., 103 (1988), pp. 319–345.

[15] ——, *Hölder estimates of solutions of singular parabolic equations with measurable coefficients*, J. Rational Mech. Anal., 118 (1992), pp. 257–271.

[16] E. DiBenedetto, Y. Kwong, and V. Vespri, *Local space-analyticity of solutions of certain singular parabolic equations*, Indiana Univ. Math. J., 40 (1991), pp. 741–765.

[17] E. DiBenedetto and Y. C. Kwong, *Harnack estimates and extinction profile for weak solutions of certain singular parabolic equations*, Trans. Amer. Math. Soc., 330 (1992), pp. 783–811.

[18] H. Engler, *An alternative proof of the Brezis-Wainger inequality*, Comm. Partial Differential Equations, 14 (1989), pp. 541–544.

[19] E. B. Fabes and N. M. Rivière, *Singular intervals with mixed homogeneity*, Studia Math., 27 (1966), pp. 19–38.

[20] A. Friedman, *Partial differential equations of parabolic type*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1964.

[21] I. Groma, F. F. Czikor, and M. Zaiser, *Spatial correlations and higher-order gradient terms in a continuum description of dislocation dynamics*, Acta Mater, 51 (2003), pp. 1271–1281.

[22] J.-S. Guo and P. Souplet, *Fast rate of formation of dead-core for the heat equation with strong absorption and applications to fast blow-up*, Math. Ann., 331 (2005), pp. 651–667.

[23] N. Hayashi and W. von Wahl, *On the global strong solutions of coupled Klein-Gordon-Schrödinger equations*, J. Math. Soc. Japan, 39 (1987), pp. 489–497.

[24] M. A. Herrero, A. A. Lacey, and J. J. L. Velázquez, *Global existence for reaction-diffusion systems modelling ignition*, Arch. Rational Mech. Anal., 142 (1998), pp. 219–251.

[25] J. R. Hirth and L. Lothe, *Theory of dislocations*, Second edition, Kreiger publishing company, Florida 32950, 1982.

[26] H. Kozono and Y. Taniuchi, *Limiting case of the Sobolev inequality in BMO, with application to the Euler equations*, Comm. Math. Phys., 214 (2000), pp. 191–200.

[27] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural′ceva, *Linear and quasilinear equations of parabolic type*, Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23, American Mathematical Society, Providence, R.I., 1967.

[28] G. M. Lieberman, *Second order parabolic differential equations*, World Scientific Publishing Co. Inc., River Edge, NJ, 1996.

[29] A. Maugeri, *A boundary value problem for a class of singular parabolic equations*, Boll. Un. Mat. Ital. B (5), 17 (1980), pp. 325–339.

[30] F. Merle and H. Zaag, *Stability of the blow-up profile for equations of the type* $u_t = \Delta u + |u|^{p-1}u$, Duke Math. J., 86 (1997), pp. 143–195.

[31] ——, *Optimal estimates for blowup rate and behavior for nonlinear heat equations*, Comm. Pure Appl. Math., 51 (1998), pp. 139–196.

[32] J. W. Mooney, *Efficient numerical algorithms for close estimation of critical parameters and quenching times in singular parabolic problems*, in Proceedings of the Second World Congress of Nonlinear Analysts, Part 2 (Athens, 1996), vol. 30, 1997, pp. 911–922.

[33] F. R. N. Nabarro, *Theory of crystal dislocations*, Oxford, Clarendon Press, 1969.

[34] T. Ogawa, *Sharp Sobolev inequality of logarithmic type and the limiting regularity condition to the harmonic heat flow*, SIAM J. Math. Anal., 34 (2003), pp. 1318–1330 (electronic).

[35] P. Quittner, P. Souplet, and M. Winkler, *Initial blow-up rates and universal bounds for nonlinear heat equations*, J. Differential Equations, 196 (2004), pp. 316–339.

[36] P. Souplet, *Blow-up behavior in nonlocal vs local heat equations*, in Evolution equations: existence, regularity and singularities (Warsaw, 1998), vol. 52 of Banach Center Publ., Polish Acad. Sci., Warsaw, 2000, pp. 221–226.

[37] G. Speranza, *Nonhomogeneous boundary value problems for the singular parabolic equation* $u_{xx} + (2\gamma/x)u_x - u_t = 0$, Matematiche (Catania), 36 (1981), pp. 136–148 (1983).

[38] H. Zaag, *On the regularity of the blow-up set for semilinear heat equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 19 (2002), pp. 505–542.